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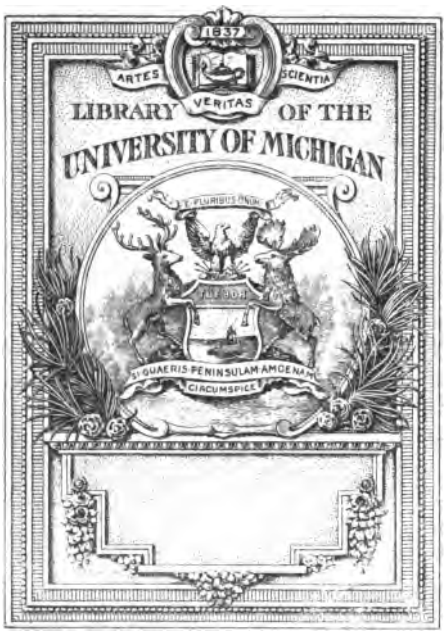

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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. MILLER, B.A.,

MATHEMATICAL MASTER, HUDDERSFIELD COLLEGE.

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CORRECTIONS AND ADDITIONS TO VOL. IV.

P. 13, line 2, see *Reprint*, Vol. III., p. 97; p. 28, line 8, *read* Vol. III.; p. 39, line 15, read $z\frac{d}{dz}$; p. 40, line 5, *after* conic *add* is; p. 43, line 34, see Salmon's *Conics*, Art. 226, ex. 3; p. 64, line 19, *for* $\frac{e_4}{2^4}$ *read* $\frac{e_n}{2^n}$, line 26, *for* $\left(e_1 + \frac{a}{b}\right)$ *read* $\left(e_1 + \frac{a_1}{b}\right)$, line 29, insert e_2, and *for* $\frac{e_n}{e_n}$ *read* $\frac{e_n}{2^n}$; p. 86, line 19, *for* (m, n, ϖ) *read* $\mathrm{F}(m, n, \varpi)$; p. 101, line 14, see *Reprint*, Vol. I., p. 50, end of Art. 6; p. 108, last line, *for* Θ *read* Θ_3.

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LIST OF CONTRIBUTORS.

ALLEN, J. R., King's College, London.
BALL, ROBERT, M.A., Trinity College, Dublin.
BICKERDIKE, C., Leeds.
BILLS, SAMUEL, Hawton, Newark-on-Trent.
BLAKEMORE, J. W. T., B.A., Stafford.
BLISSARD, Rev. J., B.A., The Vicarage, Hampstead Norris, Berks.
BOOTH, Rev. Dr., F.R.S., The Vicarage, Stone, Bucks.
BURNSIDE, W. S., B.A., Trinity College, Dublin.
CASEY, JOHN, B.A., Kingstown, Ireland.
CAYLEY, Professor, F.R.S., University of Cambridge; Corresponding Member of the Institute of France.
CLIFFORD, W. K., Trinity College, Cambridge.
COCKLE, the Hon. J., M.A., Chief Justice of Queensland; President of the Queensland Philosophical Society.
COHEN, ARTHUR, B.A.
COLLINS, MATTHEW, B.A., Dublin.
CONOLLY, E., Mountnugent, Ireland.
CONWILL, J., Leighlinbridge, Ireland.
CREMONA, Professor, University of Bologna.
CROFTON, M. W., B.A., Royal Military Academy, Woolwich.
COTTERILL, THOS., M.A., late Fellow of St. John's College, Cambridge.
DALE, JAMES, Aberdeen.
DE MORGAN, A., Professor of Mathematics, University College, London.
DE MORGAN, G. C., M.A., London.
DOBSON, T., B.A., Head Master of Hexham Grammar School.
EASTERBY, W., B.A., Grammar School, St. Asaph.
EVERETT, J. D., D.C.L., Assistant Professor of Mathematics, University of Glasgow.
FENWICK, STEPHEN, F.R.A.S., Royal Military Academy, Woolwich.
FITZGERALD, E., Bagenalstown, Ireland.
FLOOD, P. W., Ballingarry, Ireland.
FULLERTON, R., Darlington.
GARDINER, M., St. John's College, Sydney.
GODFRAY, HUGH, M.A., Cambridge.
GODWARD, W., Law Life Office, London.
GREER, H. R., B.A., Royal Military College, Sandhurst.
GRIFFITHS, J., M.A., Fellow of Jesus College, Oxford.
HANLON, G. O., Dublin.
HARLEY, Rev. R., F.R.S., Brighouse, Yorkshire.
HIRST, Dr. T. A., F.R.S., Professor of Natural Philosophy, University College, London.

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HOLDITCH, Rev. H., Caius College, Cambridge.
HOPPS, W., Hull.
HUDSON, W. H. H., M.A., St. John's College, Cambridge.
JENKINS, MORGAN, B.A., Christ's College, Cambridge.
KIRKMAN, Rev. T. P., M.A., F.R.S., Croft Rectory, near Warrington.
KITCHIN, Rev. J. L., M.A., Loughborough.
LAVERTY, W. H., Queen's College, Oxford.
LEVY, W. H., Shalbourne, Berks.
MANNHEIM, M., Professeur à l'Ecole Polytechnique, Paris.
MCCOLL, HUGH, Collège Communal, Boulogne.
MCCORMICK, E., Hereford.
MCDOWELL, J., M.A., F.R.A.S., Pembroke College, Cambridge.
MERRIFIELD, C. W., F.R.S., Principal of the Royal School of Naval Architecture, South Kensington.
MILLER, W. J., B.A., Huddersfield College.
MURPHY, HUGH, Killeshandra, Ireland.
NELSON, R. J., M.A., London.
O'CAVANAGH, PATRICK, Dublin.
PROUHET, E., Répétiteur à l'Ecole Polytechnique, Paris.
PURKISS, the late H. J., B.A.
RADCLIFFE, J. E.
RENSHAW, A., Sherwood Rise, Nottingham.
ROBERTS, Rev. W., M.A.
ROBERTS, SAMUEL, M.A., London.
ROSOLIS, E. L.
RUTHERFORD, Dr., F.R.A.S., Woolwich.
SADLER, G. T., F.R.A.S., London.
SALMON, Rev. Dr., F.R.S., Trinity College, Dublin.
SPOTTISWOODE, W., F.R.S., London.
STANLEY, ARCHER.
SYLVESTER, Professor, F.R.S., Royal Military Academy, Woolwich; Corresponding Member of the Institute of France.
TAYLOR, C., M.A., St. John's College, Cambridge.
TAYLOR, H. M., B.A., Vice-Principal of the Royal School of Naval Architecture, South Kensington.
TAYLOR, J. H., B.A.
TAYLOR, JOHN, Salop.
THOMSON, F. D., M.A., Exeter.
TODHUNTER, I., M.A., F.R.S., St. John's College, Cambridge.
TORRY, A. F., M.A., St. John's College, Cambridge.
TOWNSEND, Rev. R., M.A., Trinity College, Dublin.
TUCKER, R., M.A., Mathematical Master, University College School, London.
WALMSLEY, J., Northampton.
WARREN, R., B.A., Trinity College, Dublin.
WATSON, STEPHEN, Haydonbridge.
WHITWORTH, W. Allen, M.A., Professor of Mathematics and Natural Philosophy, Queen's College, Liverpool.
WILKINSON, T. T., F.R.A.S., Burnley.
WILSON, J., 45th Regiment, Colaba Camp, Bombay.
WILSON, J. M., M.A., Rugby School.
WOOLHOUSE, W. S. B., F.R.A.S., &c., London.
WRIGHT, Rev. R. H., M.A., Head Master of Ashford Grammar School.

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1493. If a conic have double contact with two fixed conics, find the envelope of its chords of intersection with another fixed conic having double contact with one of the two fixed conics. 40

1497. (1.) Given three points by equations of the form $lx + my + nz = 0$; prove that the area of the triangle contained by them is

$$(l_1 m_2 n_3) \div (l_1 + m_1 + n_1)(l_2 + m_2 + n_2)(l_3 + m_3 + n_3),$$

that of the triangle of reference being unity.
(2.) Also, if (123) denote the area of the triangle contained by the points 1, 2, 3, and so on, prove that
(123) (456) $\equiv$ (156) (423) + (164) (523) + (145) (623). ... 53

1551. Using the notation of Spence's Transcendents, viz.,

$$L^n(1+x) = \frac{x}{1^n} - \frac{x^2}{2^n} + \frac{x^3}{3^n} - \frac{x^4}{4^n} + \text{ &c., prove that}$$

$$L^2(1+x) + L^2\left(\frac{1}{1+x}\right) = \tfrac{1}{2}\log^2(1+x); \text{ also that}$$

$$L^2\left(\frac{1}{1+x}\right) + L^2\left(\frac{x}{1+x}\right) = \log^2(1+x) - \log x \log(1+x) - \tfrac{1}{6}\pi^2. \quad 96$$

1586. If a straight line meet the sides BC, CA, AB of a triangle in P, Q, R respectively, the tangents at O to the conics OAPBQ, OAPCR form with the straight lines OA, OP a harmonic pencil. Also, if a fourth harmonic CK be drawn to the pencil CA, CO, CP, and a fourth harmonic RK to the pencil RA, RO, RP, these will intersect upon the tangent at O to the conic OAPBQ; and if a fourth harmonic be now drawn to KC, KO, KR, it will be *invariable* with respect to the three conics OAPBQ, OBQRC, OCRAP. 44

1629. To investigate the geometrical signification of e, the base of the Naperian logarithms. 41

1672. Give simple geometrical constructions for (1) inscribing the greatest triangle in an ellipse, and (2) circumscribing the least triangle about an ellipse. 91

1675. If a triangle abc be the reciprocal of ABC in respect of a parabola whose parameter is $4m$; and if n_1, n_2, n_3 be the normals at the vertices of diameters through ABC; then

$$\frac{(\text{area of } abc)^2}{bc \cdot ca \cdot ab} = \frac{2m^2}{n_1 n_2 n_3}(\text{area of ABC}). \quad 90$$

1677. Prove that a chord of constant length, drawn through either point of contact of a circle which has double contact with a fixed conic, envelopes a confocal conic. 42

1684. Prove by inversion the leading properties of the Lemniscate of Bernoulli, including those of its osculating circle and Saladini's property that it is the curve down which a particle will descend through any arc in the same time as it will take to descend through the chord of the arc. Show also that it possesses, in a modified way, anharmonic properties similar to those of the conic sections. 47

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1692. Prove that, n being any positive integer,

$$\frac{2^{2n}}{\lfloor 2n+1} = \frac{1}{\lfloor 2n+1} + \frac{1}{\lfloor 2n-1}\frac{1}{\lfloor 2} + \frac{1}{\lfloor 2n-3}\frac{1}{\lfloor 4} + \ldots\ldots\ldots\ldots + \frac{1}{\lfloor 2n},$$

$$\frac{2^{4n-2}}{\lfloor 4n} = \frac{1}{2}\frac{1}{(\lfloor 2n)^2} + \frac{1}{\lfloor 2n-2}\frac{1}{\lfloor 2n+2} + \frac{1}{\lfloor 2n-4}\frac{1}{\lfloor 2n+4} + \ldots\ldots + \frac{1}{\lfloor 4n},$$

$$= \frac{1}{\lfloor 2n-1}\frac{1}{\lfloor 2n+1} + \frac{1}{\lfloor 2n-3}\frac{1}{\lfloor 2n+3} + \ldots\ldots\ldots\ldots + \frac{1}{\lfloor 4n-1},$$

$$\frac{2^{4n}}{\lfloor 4n+2} = \frac{1}{2}\frac{1}{(\lfloor 2n+1)^2} + \frac{1}{\lfloor 2n-1}\frac{1}{\lfloor 2n+3} + \ldots\ldots\ldots\ldots + \frac{1}{\lfloor 4n+1},$$

$$= \frac{1}{\lfloor 2n}\frac{1}{\lfloor 2n+2} + \frac{1}{\lfloor 2n-2}\frac{1}{\lfloor 2n+4} + \frac{1}{\lfloor 2n-4}\frac{1}{\lfloor 2n+6} + \ldots + \frac{1}{\lfloor 4n+2}$$ 62

1695. If u_m be a homogeneous function of the degree m in any number of independent variables, prove that

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\ldots\right)\log u_m, \text{ identically;}$$

$$\left(x^2\frac{d^2}{dx^2} + y^2\frac{d^2}{dy^2} \text{ \&c.} + 2xy\frac{d}{dx}\cdot\frac{d}{dy} \text{ \&c.}\right)\log u_m = -m;$$

and continue the theorem. 39

1696. Supplemental chords of an ellipse are produced to meet the tangents at the extremities of the corresponding diameter: prove that the rectangle contained by the intercepts on the tangents is constant. 43

1697. If two circles touch a straight line a, and have a different kind of contact with two other straight lines b and c, prove that the middle point of the distance of a from the point of intersection of b and c lies in a straight line with the centre of one of the two circles and the point where the other circle touches a.

N.B.—The contact of two circles with two straight lines is said to be like, or of the same kind, when the two circles are on the *same side* of *both* lines, or else on *different sides* of *both* lines; but if the two circles lie on the same side of one of the two straight lines, and on different sides of the other straight line, then the contact of the two circles with the two straight lines is said to be unlike, or of different kinds. 98

1698. Prove that the foot of a perpendicular on a common tangent (a) to two circles, drawn from the point of intersection of the two contrary common tangents (b and c) to those circles, lies in a straight line with the centre of one of the circles and the reflexion of the centre of the other with respect to the tangent (a). 98

1704. From any point in a straight line perpendiculars are drawn on the sides of a triangle, and the feet of these perpendiculars joined; find the locus of the centre of gravity (or *centroid*) of the triangle thus formed. 60

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1707. If A, B, C are the distances of a line from the angles of a triangle which it cuts in the points d, e, f respectively; then

$$\frac{A\,(B-C)}{ef} = \frac{B\,(C-A)}{fd} = \frac{C\,(A-B)}{de} = \frac{(B-C)\,(C-A)\,(A-B)}{2k},$$

k being the area of the triangle. The envelope of the lines in which $\lambda \,.\, ef+\mu \,.\, fd+\nu \,.\, fe = 0$ is a parabola: λ, μ, ν being as the coordinates of its axis. This equation can be written in such shapes, that an orthogonal substitution gives the envelope of the tangent at the vertex, and their form shows that they are also the envelope of the asymptotes of the rectangular hyperbolas circumscribing the triangle. The intercept of such a tangent between two sides of the triangle is equal to the projection of the remaining side upon it. 45

1709. 1. Let $f(x)$ be any function of x, and let $f(a, b)$ express (according as a is greater or less than b) a superior or inferior limit to $f(x)$ for all values of x between a and b; show that if $f(a, b)$ has the same sign as $b-a$, no real root of the equation $f(x) = 0$ exists between a and b.

2. Let $f(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots\ldots + A_1 x + A_0$, n being a positive integer. Show that, when a and b are positive, $f(a, b)$ may be found by the following process. According as A_n is positive or negative, multiply it by a or b; add the product to A^{n-1}, and represent the result by k_1. According as k_1 is positive or negative, multiply it by a or b; add the product to A_{n-2} and represent the result by k_2. Proceeding in this way till $k_1, k_2, k_3 \ldots\ldots k_n$ are all found, we shall have $f(a, b) = k_n$. 57

1710. Trace the curve $y^4 - 2y^2zx - z^4 = 0$, where the coordinates are such that $x+y+z = 0$ is the line *infinity*. 17

1711. To a uniform beam supported at its extremities is suspended a weight equal to that of the beam: show that, if the point of suspension is nearer to the centre than to an end of the beam, the moment of strain will be greatest at that point, but otherwise greatest at a point at the same distance from the centre, as the point of suspension is from the end of the beam. 29

1712. Let $f(x)$ be any rational integral function of (x) of the degree n, and $f'(x)$ the first derived function; then will $f'(x_1) \,.\, f'(x_2) \ldots\ldots f'(x_n) = a^{2-n} n^n \Delta$, where a is the coefficient of the highest power of x in $f(x)$, Δ is the discriminant, and $x_1, x_2 \ldots\ldots x_n$ are the roots of $f(x) = 0$. 50

1715. Let P be the point of intersection of the perpendiculars of any triangle ABC, O the centre of the circumscribed circle, M the middle point of PO and O′ the centre of any one of the escribed circles of the triangles ABC, BPC, CPA, APB. Prove that the nine-point circle of the above triangle touches the common tangent of the circle (O′) and the circle described on MO′ as diameter. 60

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1717. Show how the locus of the foci of conics, subject to four conditions, may be found in the following cases:—(1.) When each conic of the system touches three lines, and satisfies any fourth condition. (2.) When each conic passes through three fixed points, and satisfies any fourth condition. (3.) When a given triangle is self-conjugate with regard to each conic of the system, which also satisfies any fourth condition. 18

1721. Find the locus of a point such that, if perpendiculars are drawn therefrom on the sides of a quadrilateral, a circle will pass through the feet of these perpendiculars; and show geometrically how to find points on the locus; also determine the quadrilateral in order that the locus may be a conic. ... 40

1724. The equations of three conics being given in the following forms: viz.,

$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0$, a_2x^2+ &c. $=0$, a_3x^2+ &c. $=0$,

where $x+y+z+w\equiv 0$, show that a straight line $(\xi x+\eta y+\zeta z+\omega w=0)$ will be cut in involution by them if

$$\Sigma\cdot\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}\cdot(\xi-\eta)\,(\xi-\zeta)\,(\xi-\omega)\cdot(\text{to four terms})=0. \quad 52$$

1726. Let O_1, O_2, O_3 be the centres of the escribed circles having internal contact with the sides BC, CA, AB respectively of the triangle ABC, and P any other point in the same plane; then (1) if R_1, R_2, R_3 be the intersections of AP, BP, CP with BC, CA, AB, respectively, the triad (O_1R_1, O_2R_2, O_3R_3) will be concurrent; and (2) if S_1, S_2, S_3 be the intersections of O_1P, O_2P, O_3P with BC, CA, AB, respectively, the triad (AS_1, BS_2, CS_3) will be concurrent. 93

1727. If the middle points of the sides of any triangle be joined, and the middle points of the sides of this new triangle also joined, and so on *ad infinitum*; the limit to which these triangles tend is a point, and the sum of the squares on the lines drawn therefrom to the angles of all the inscribed triangles is one-third of the sum of the squares on the lines drawn from the same point to the angles of the original triangle. 76

1732. Prove that the characteristics (see Question 1573) of a system of conics, satisfying four conditions, remain unaltered when, in place of passing through a given point, each conic is required to divide a given finite segment harmonically. 56

1733. To find the area of a triangle the equations of whose sides in trilinear coordinates are

$l_1\alpha+m_1\beta+n_1\gamma=0$, $l_2\alpha+m_2\beta+n_2\gamma=0$, $l_3\alpha+m_3\beta+n_3\gamma=0$. 53

1737. Solve the equations

$\alpha xy+\alpha' xz+a=0$, $\beta yz+\beta' yx+b=0$, $\gamma zx+\gamma' zy+c=0$. ... 50

1739. Let $f(x)=A_nx^n+A_{n-1}x^{n-1}+\ldots+A_1x+A_0$. Let $f_1(x)=A_{n-1}x^{n-1}+2A_{n-2}x^{n-2}+\ldots+(n-1)A_1+nA_0$. Form $f_2(x)$ from $f_1(x)$ in the same manner, and so on to $f_n(x)$. Let a

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and b be any two quantities *of the same sign*, of which a is greater numerically than b. Show that if the series of functions $f(x)$, $f_1(x)$, $f_2(x) \ldots f_n(x)$, give p changes of signs when $x = a$, and q changes when $x = b$, the number of real roots of $f(x) = 0$ between a and b cannot exceed $p-q$. 102

1741. Let P be the point of intersection of the perpendiculars, and G the centre of gravity, of any triangle ABC; prove that the nine-point circles of all the triangles that can be formed from the points A, B, C, P, G, together with the maximum ellipse that can be inscribed in the triangle ABC, all pass through a common point, viz., the centre of the equilateral hyperbola, passing through A, B, C, P, G. 89

1743. If P, Q be two points in the path of a projectile; PT, QT the tangents at P, Q; and α, β, γ the angles made with the horizon by PT, TQ, QP; prove geometrically that

$$\tan\alpha + \tan\beta = 2\tan\gamma.$$

2. Hence show that if a particle be projected from one end of the horizontal base of an isosceles triangle so as exactly to shoot down the opposite side; and if α be the angle of projection, γ the angle at the base of the triangle; then

$$\tan\alpha = 3\tan\gamma.$$

3. Prove that the velocities at P, Q are proportional to PT, QT. 100

1744. It is required to find (x_1, y_1, z_1), functions of (x, y, z), such that we may have identically

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1} = \frac{x^3+y^3+z^3}{xyz}. \quad \ldots\ldots\ldots 32$$

1749. AB is a diameter of a circle, O its centre, and P a point in OA, such that $OP = \frac{1}{3}OA$. Through P any straight line QPR is drawn, meeting the circumference in Q, R; and the straight line passing through Q, O is produced to meet that through B, R in S.

If $\angle OSB = \psi$, and $\angle OBS = \phi$, prove that $\frac{1-\sin\psi}{1+\sin\psi} = \left(\frac{1-\sin\phi}{1+\sin\phi}\right)^3$. 20

1750. Given four straight lines whose equations are connected by the syzygy $x+y+z+w=0$; show that the straight lines

$$lx+my+nz+sw=0, \quad \lambda x+\mu y+\nu z+\sigma w=0,$$

will be conjugates in respect of any conic touching (x, y, z, w), if $(l\mu+\lambda m)+n\sigma+\nu s) = l\nu+\lambda n)+(su+\sigma m) = (l\sigma+\lambda s)+(m\nu+\mu n)$. Show also that if a quadrangle be formed from the quadrilateral by taking the pole of each line in respect of the triangle formed by the other three, then the relation between the two figures will be reciprocal; and if two straight lines be conjugates in respect of any conic inscribed in the quadrilateral, their poles in respect of the common connector-triangle will be conjugates in respect of any conic circumscribing the quadrangle. 63

1751. Let ABCD be any quadrilateral. Construct, as shown in the figure, the points F, G, H, I: in BC find a point Q such that $\frac{BG}{BC} \cdot \frac{CQ}{GQ} = \frac{1}{\sqrt{2}}$; and complete the construction as

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shown in the figure. Show that an ellipse may be drawn passing through the eight points F, G, H, I, α, β, γ, δ, and having at these points respectively the tangents shown in the figure. 38

1752. Prove that any proper fraction may be expanded under the form $\frac{e_1}{2} + \frac{e_2}{2^2} + \frac{e_3}{2^3} + \ldots\ldots\frac{e_n}{2^n} + \ldots\ldots$ where the e's are all either zero or unity, and moreover become periodic after a certain point in the series; and show thereby that a calculating machine capable of multiplying and extracting square roots, will serve for the extraction of any root whatever. 64

1753. The tangential equation of the surface of the centres of curvature of an ellipsoid being

$$(\xi^2+\upsilon^2+\zeta^2) = \left(\frac{\xi^2}{a^2}+\frac{\upsilon^2}{b^2}+\frac{\zeta^2}{c^2}\right)(a^2\xi^2+b^2\upsilon+c^2\zeta^2-1);$$

show that any two parallel tangent planes being drawn to the surface of centres and to the ellipsoid, the difference of the squares of the coincident perpendiculars let fall upon them from the common centre is always equal to the square of the coinciding semidiameter of the ellipsoid 115

1754. If a triangle and its nine-point circle are drawn; in the parabola which touches two sides and the perpendiculars to them through the opposite angles, the focus, the directrix, and the tangents to it from the centre of the circle are given by the figure..... 114

1756. If normals drawn to a central quadric at the points (1), (2), (3), (4), (5), (6) pass through a point, and r_{12} denote the length of the semi-diameter parallel to the chord (1, 2); prove that $r_{12}^2 + r_{34}^2 + r_{56}^2 = a^2+b^2+c^2$, where a, b, c are the semi-axes of the quadric 99

1760. Let each angular point of any triangle ABC be joined with each of two given points: and let α, β, γ; α', β', γ' denote the six points of intersection of the joining lines with the opposite sides of the triangle. It is required to prove that the conic which touches the sides of the two triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ touches also the two axes of homology of the triangles ABC, $\alpha\beta\gamma$, and ABC, $\alpha'\beta'\gamma'$. Show also that the triangle ABC is self-conjugate with respect to this conic. What is the reciprocal of this theorem? 65

1761. Let $f(x) = 0$ be any equation of the nth degree. Let $f^r(x)$ be the rth derived function of $f(x)$. Let it be known that $f^r(x) = 0$ has only one root between the two quantities a and b. Show that by giving a more general expression to the trial fraction in Horner's method of approximation we can simultaneously approximate as nearly as we please to the root of $f^r(x) = 0$, separate any two roots of $f^{r-1}(x) = 0$ that may exist between a and b, and obtain information as to the number of roots of $f^{r-2}(x) = 0$, $f^{r-3}(x = 0, \ldots f^1(x) = 0$, $f(x) = 0$ passed over in the operation. Show also

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that the same process will enable us to approximate simultaneously to any maximum or minimum value of $f(x)$, and to the value of x, which makes $f(x)$ a maximum or minimum. 102

1765. Two weights are attached by short strings of given length to the surface of a cylinder; determine the position of equilibrium of the cylinder when resting on a horizontal plane, the weights resting on the surface of the cylinder 86

1766. To find the condition necessary in order that one of two homofocal ellipses should be inscribed in a triangle, and the other circumscribed about it..... 49

1771. Given a circle and a line, it is required to find a parabola, having the line for its directrix, and the circle for a circle of curvature..... 70

1773. A traveller starts from a point on a straight river, and travels in a random direction a certain distance in a straight line. Having quite lost his way, he starts again at random the next morning, and travels the same distance; what is the chance of his reaching the river again in the second day's journey? 71

1774. If the ends of a fixed diameter of a circle be joined to the ends of a chord variable in position, but of given magnitude, the locus of the centre of the circle circumscribing the triangle formed by the chord and the segments of the joining lines outside the circle, is a circle concentric with a given one. 72

1775. If a straight line meet the faces of the tetrahedron ABCD in the points a, b, c, d, respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis. 65

1780. Find the locus of a point whose polar, in reference to a given conic, cuts off a given area from the axes. 68

1781. From a point P in the produced major axis of an ellipse to draw a line PQR, cutting the curve in Q and R, so that if Q and R be joined with the centre O of the ellipse, the triangle OQR may be a maximum..... 69

1784. If from any given point in an ellipse there be drawn an ordinate to either axis, and if from the point where this meets the curve again a diameter be drawn, and to this diameter another ordinate be applied from the given point; then the point in which this ordinate meets the ellipse shall be in the circumference of the circle of curvature at the given point. 97

1786. If $4a$ be the parameter of a parabola, S_y the length (measured from the vertex) of an arc the ordinate of whose extremity is y, and R_y the radius of curvature at the extremity, prove that $S_{4a} - 3S_a = R_a$. 69

1789. Let abc be the triangle formed by the three diagonals AA', BB', CC' of a quadrilateral: and let Aa, B'b, C'c meet A'BC in $\alpha_1, \alpha_2, \alpha_3$; A'$a$, B'$b$, C$c$ meet AB'C in $\beta_1, \beta_2, \beta_3$; A'$a$, B'$b$, C$c$,

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meet ABC′ in $\gamma_1, \gamma_2, \gamma_3$; A$a$, B$b$, C$c$, meet A′B′C in $\delta_1, \delta_2, \delta_3$; then $\alpha_2, \beta_2, \gamma_2, \delta_2$, are the points of contact of the "*Critical* Inscribed Conic" of the quadrilateral CA′C′A (this may be taken as the *Definition* of the "*Critical* Inscribed Conic"), and Professor Cayley has given in Quest. 1751 (*Reprint*, Vol. IV., p. 38), a construction for finding four other points on the conic and the tangents thereat. It is required to show that, by the ruler alone, *eight* other points on this conic, and the tangents thereat, may be found by the following

Construction.—Let $\alpha_2\beta_2$ meet C′A′ in p; join $p\gamma_3$ cutting Cδ_2 in P; then P is a point on the curve, and Pp the tangent at P. Similarly, *seven* other points may be found, and the tangents constructed thereat 87

1790. (1.) If a set of six points be respectively represented by the six permutations of $a : b : c$, show that they lie in a conic, and write down its equation.

(2.) *Hence* prove that if AB, BC, CA be three real lines containing the nine points of inflexion of a cubic curve having an oval, the pairs of tangents drawn to the oval from A, B, C will meet it in six points lying in a conic 81

1795. (1) Let P be the point in a homogeneous triangular lamina ABC at which the sides subtend equal angles. Show that if the lamina be placed in a smooth prolate spheroid whose long axis is vertical, it will rest in equilibrium when the point P coincides with the lower focus of the spheroid.

(2) If the lamina be not homogeneous, and its centre of gravity be given, construct for the corresponding position of the point P. 115

1807. From the ends of a diameter of a semicircle, two persons walk in random directions within the semicircle; find the chance of their paths intersecting inside the circle. 92

1811. (α). Prove that the chance of three points taken at random within a circle or sphere forming the vertices of a triangle of any prescribed form is the same as if two of the points are restrained to move on the circumference or surface, with a likelihood varying as the square of their distance apart.

(β). By the aid of this principle show that the chance of a triangle whose angular points are taken at random within a circle being acute is

$$\int_0^{\frac{1}{4}\pi} (3\sin^3\theta\cos\theta + \theta\sin^2\theta - \tfrac{1}{2}\pi\sin^4\theta)\,d\theta + \int_0^{\frac{1}{4}\pi} \sin^2\theta\,d\theta, \text{ or } \frac{4}{\pi^2} - \frac{1}{8};$$

and that the chance of a triangle whose angular points are taken at random within a sphere being obtuse is

$$\int_0^{\frac{1}{4}\pi} \sin^3\cos\theta\,(6 - 6\sin\theta - 3\cos\theta + 4\sin^3\theta + \cos^3\theta)\,d\theta \text{ or } \frac{37}{70}.$$

(γ). Find also the chance that no angle of the triangle will exceed any magnitude α between $\frac{1}{3}\pi$ and π, distinguishing between the case where α is greater and that where it is less than $\frac{1}{2}\pi$. 111

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1812. Find the envelope of a series of circles which touch a given straight line and have their centres in the circumference of a given circle. 107

1814. A triangle is inscribed in an ellipse and envelopes a confocal ellipse: prove that the points of contact lie on the escribed circles..... 108

1816. Express the roots of the equation

$$(ae-4bd+3c^2)(ax^4+4bx^3+6cx^2+4dx+e)^2-3\{(ac-b^2)x^4+2(ad-bc)x^3 + (ae+2bd-3c^2)x^2+2(be-cd)x+(ce-d^2)\}^2=0,$$

in terms of the roots $\alpha, \beta, \gamma, \delta$ of $x^4+4bx^3+6cx^2+4dx+e=0$. 108

1823. The conicoids which pass through six fixed points in space intersect any plane in a series of conics having a common self-conjugate quadrilateral. *Any four* conics have a common self-conjugate quadrilatral. (DEF.—A quadrilateral is *self-conjugate* in respect of a conic which divides its diagonals harmonically) 109

1828. Exhibit the expression

$$(\beta-\gamma)^2(\beta+\gamma-2\alpha)+(\gamma-\alpha)^2(\gamma+\alpha-2\beta)+(\alpha-\beta)^2(\alpha+\beta-2\gamma)$$

(1) as the product of three factors, (2) as the sum of two cubes 110

1829. Two points being taken at random within (1) a circle or (2) a sphere, what is the probability that their distance apart is less than a given line? 77

1847. Prove that the method given by Tschirnhäusen for removing the second and fourth terms from the binary quartic $(a, b, c, d, e)(x, y)^4 = 0$ depends ultimately on the solution of the equation $\lambda^3 - I\lambda + 2J = 0$,

where $I = ae-4bd+3c^2$, and $J = ace+2bcd-ad^2-eb^2-c^3$... 111

Unsolved Questions.

1480. (Proposed by Professor SYLVESTER.)—Prove that if through the middle point of either diagonal of any of the three quadrilateral faces of a tetrahedral frustum, and the middle points of the two edges which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will touch the same cone.

1481. (Proposed by Professor HIRST.)—Find the envelope of a conic which circumscribes a given triangle, and is cut harmonically by two fixed straight lines.

N.B.—A conic is said to be cut harmonically in two pairs of points, when the tangents at those points cut every other tangent harmonically, or, what amounts to the same thing, when the connectors of those points with any other point on the conic form a harmonic pencil.

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1495. (Proposed by HUGH GODFRAY, M.A.)—Show that $\frac{1}{2}n(n-1)(n-2)$ points can always be so arranged in a plane that they shall be situated by eights in $\frac{1}{24}n(n-1)(n-2)(n-3)$ circles.

1496. (Proposed by MATTHEW COLLINS, B.A.)—Prove that a triangular pyramid whose vertices are ABCD, and a parallelepiped formed from it as follows, have the same centre of gravity; viz., through any point in each of the opposite edges AB and CD draw straight lines parallel to the other edge, we thus get two parallel plane faces of the parallelepiped; two other parallel faces of it are similarly obtained from the opposite edges AC and BD; and the third pair of faces are obtained from the remaining two opposite edges AD and BC.

1503. (Proposed by Professor SYLVESTER.)—A table has n holes bored in its rim, into which ν pegs are to be inserted at random, ν being not greater than $\frac{1}{2}n$. Show that the probability of there being no two pegs without one or more unoccupied holes between them will be equal to $\frac{\pi(n-\nu)\cdot\pi(n-\nu-1)}{\pi(n-1)\cdot\pi(n-2\nu)}$, and, if ν is given, approaches to certainty as n becomes indefinitely great.

[$\pi(n)$ here denotes the product $1.2.3....n$.]

1504. (Proposed by Professor SYLVESTER.)—Trace the two sextic curves

$$y+27x^3-28x^3=x^2\sqrt{(72x-80x^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{y^{\frac{1}{2}}}{6\sqrt{3}}=(1-x)+(1-x)^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and prove that they intersect in two real points and touch in a third point which are to be found.

1511. (Proposed by Professor SYLVESTER.)—At a dancing and card party lately held at the Palazzo Finto, l ladies and g gentlemen sat down miscellaneously at a table to play a round game; when, on its being announced that more dancers were wanted, every cavalier who had a lady sitting next to him on his right hand, gave her his arm to lead her into the ball-room. Prove that the number of couples which there is the *greatest* probability of having been thus formed is the integer part of the quotient of lg by $(l+g+1)$, if the division leaves a *remainder*; but if there is *no remainder*, then indifferently that number or the number next below it.

1515. (Proposed by the late H. J. PURKISS, B.A.)—Find the conditions that the general equation of the second degree in tetrahedral coordinates may represent a surface of revolution.

1554. (Proposed by Professor CAYLEY.)—Show that in the ellipse and its circles of maximum and minimum curvature respectively, the semi-ordinates through the focus of the ellipse are,

for the circle of maximum curvature, $y=a(1-e)(1+2e)^{\frac{1}{2}}$;

for the ellipse, $y=a(1-e^2)$;

for the circle of minimum curvature, $y=\dfrac{a\left\{1-e^2+e^4\right)^{\frac{1}{2}}+e^2\right\}}{(1-e^2)^{\frac{1}{2}}}$;

and that these values are in the order of increasing magnitude.

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1587. (Proposed by the Rev. T. P. KIRKMAN, F.R.S.)—Apollo and the Muses accepted the challenge of Jove, to vary the arrangement of themselves on their fixed and burnished couches at his evening banquets, till every three of them should have occupied, once and once only, every three of the couches, in every and any order. In how many days, and how many ways, did they accomplish the feat, keeping one arrangement of themselves through all the solutions? Required two or more of those solutions, clearly indicated, so as to save space, by cyclical operations.

1604. (Proposed by Professor SYLVESTER.)—Find the area, centre of gravity, and principal axes of the segment cut off from any conic by any straight line; the whole expressed in trilinear coordinates.

1611. (Proposed by Dr. BOOTH, F.R.S.) — The inverse curve of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, the centre being the pole and $k^2 = ab$, is $a^2y^2 + b^2x^2 = (x^2 + y^2)^2$. Rectify this curve, and show that its length is equal to that of a spherical ellipse, the symmetrical intersection of a sphere whose radius is $\sqrt{(a^2 + b^2)}$ with a right cylinder standing on the original ellipse.

1625. (Proposed by Professor CREMONA.)—If from a point on a cubic four straight lines be drawn to touch the curve elsewhere, three of the latter will be the polar lines of the point of contact on the fourth; relative, respectively, to the three cubics whose common Hessian is the given one.

1634. (Proposed by W. A. WHITWORTH, M.A.)—Required the geometrical *interpretation* of the following property:—

If $(\alpha', \beta', \gamma')$ be any point on a conic whose equation in trilinear coordinates is $f(\alpha, \beta, \gamma) = 0$, then the condition that $f(\alpha + i\alpha', \beta + i\beta', \gamma + i\gamma')$ should be a real quantity, being expressed algebraically, is the equation to the tangent to the conic at the point $(\alpha', \beta', \gamma')$.

[i denotes, as usual, the square root of negative unity, $\sqrt{(-1)}$.]

1637. (Proposed by Professor HIRST.)—If a, b, c be fixed points on cubic, what will be the envelope of the line joining a variable point v thereon to the opposite of the four points a, b, c, v?

1640. (Proposed by A. RENSHAW.)—Given four points A, B, C, D in a plane. Find the locus of a fifth point P in the plane, such that $PA \pm PB = PC \pm PD$. What will the locus be when the system is not restricted to a plane?

1641. (Proposed by E. MCCORMICK.)—An ellipse is placed with its major axis vertical; find geometrically the straight line of quickest descent from the upper focus to the curve.

1674. (Proposed by J. W. T. BLAKEMORE, B.A.)—A cylinder filled with fluid is closed at both ends, and then suspended from a point in the rim of one end; prove that the direction of the resultant pressure on the curved surface, and the axis of the cylinder, make equal angles with the vertical and horizontal respectively.

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

On the Envelope in Question 1679.

By the Rev. R. Townsend, M.A.

Nearly all the interesting results enunciated by Steiner in his paper on this Envelope (see the *Educational Times* for June) may be readily deduced from the two following properties of the hypocycloid of three cusps:

(I.) *The chord intercepted by the inscribed circle on any tangent to the curve is trisected externally at its point of contact with the curve.*

(II.) *The two arcs into which the inscribed circle is divided by any tangent to the curve are trisected internally at their points of contact with the corresponding branches of the curve.*

These may be proved as follows:—Let A, B, C (Fig. 1) be the positions of the three cusps on the circumscribed circle; E, F, G those of the points of contact of the three opposite branches with the inscribed circle; O the common centre of both circles; L any point on the former circle; P the corresponding point on the curve; and XZ the chord intercepted by the latter circle on the tangent at P, which, being perpendicular to PL, passes, as is well known, through the point of contact X of the generating and inscribed circles. Then, since, from the equality of the two latter circles, chord XZ = chord XP, there-

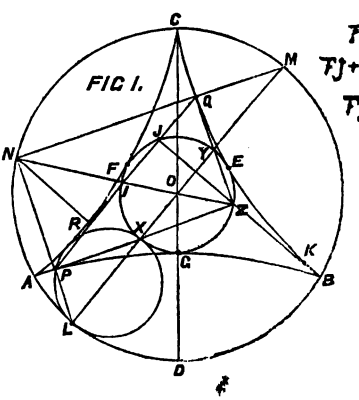

FIG I.

$FJ + YE = FI + XG$

$FJ + \frac{1}{3}YZ = FI + \frac{1}{3}XZ$

$FJ - FI = \frac{1}{3}(XZ - YZ)$

$= \frac{1}{3}(XJ - JY)$

$= \frac{1}{3}IJ$

$\therefore FI = \frac{1}{2}FJ$

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fore length PZ = 2 length PX, which proves (I.); and since, from the equality of the two arcs XP and LD, and consequently of the two arcs XZ and LD, arc XZ : arc XG = arc LD : arc XG = rad. OD : rad. OG = 3 : 1, therefore (II.) follows; the trisection of the arc XFZ at F being, of course, necessarily involved in that of the arc XGZ at G.

N.B.—By virtue of (I.), the point X was termed by Steiner the *centre* of the tangent PZ; and the point Z, for distinction, the *vertex* of the same. These terms are convenient, and may be employed when required.

It follows of course, conversely, from the above, that, *If a variable arc* XGZ *of a fixed circle have a fixed point of internal trisection* G, *its chord* XZ *envelopes a tricusped hypocycloid, which touches the circle at* G, *and at the two other points* E *and* F *which with* G *trisect the circumference; and the chord itself, in every position, is trisected externally at its point of contact with the envelope.* These are evident from the direct properties.

Cor. 1.—If a chord ZX of the inscribed circle touch any branch of the curve at P, the perpendicular chord ZY at its vertex Z touches another of its branches at Q, and the line PQ, connecting the two points of contact P and Q, touches the third branch at R. Also, the length of the connecting line PQ is constant, whatever be the position of the first tangent ZX, and equal to twice the diameter of the inscribed circle XY; and the three normals to the curve at the three points of contact P, Q, R intersect at a common point N on the circumscribing circle, diametrically opposite to the position of Z.

For, if the arc XGZ intercepted by XZ be trisected, as it is by hypothesis, at G, then manifestly the arc YEZ intercepted by YZ is trisected at E; and as the line PQ joining the points of contact P and Q of the two tangents ZX and ZY is then parallel to the diameter XY and at double its distance from the point Z, the arc IFJ which it intercepts is trisected at F; which proves the first part. And, as the two normals PL and QM, at the two points P and Q of the curve, form with the two tangents PZ and QZ at those points a rectangle PZQN, whose diagonals bisect each other, therefore their point of concurrence N is on the circumscribing circle and diametrically opposite to Z; and, the two triangles NIR and ZIJ being equal in all respects, as the angle at J is a right angle, so also is the angle at R, or the line NR is the normal at the point of contact R of the third tangent PQ, whose length intercepted between the two points P and Q is constant and equal to twice that of the diameter XY; which proves the remainder.

N.B.—Since PNQ is a right angle, it follows that the two points L and M on the circumscribing circle, corresponding to the points of contact P and Q of the two rectangular tangents ZX and ZY, are diametrically opposite points of that circle.

As the tricusped hypocycloid manifestly admits in no direction of two real tangents parallel to each other, it follows consequently from the above, that *In a tricusped hypocycloid, when two tangents intersect at right angles,* (*a*) *they belong to different branches,* (*b*) *they correspond to diametrically opposite points on the circumscribed circle,* (*c*) *they intersect on the inscribed circle,* (*d*) *their chord of contact is of constant length,* (*e*) *and is a tangent to the third branch,* (*f*) *their corresponding normals intersect on the circumscribed circle,* (*g*) *and are concurrent with the normal corresponding to their chord of contact.*

It follows from this last, of course, conversely, that, in a tricusped hypocycloid, (*a*) the segment intercepted by any two of the three branches on any tangent to the third is of constant length, (*b*) the tangents to those branches at its extremities intersect at right angles on the inscribed circle, (*c*) the normals corresponding to the three tangents concur to a point on the circumscribed circle.

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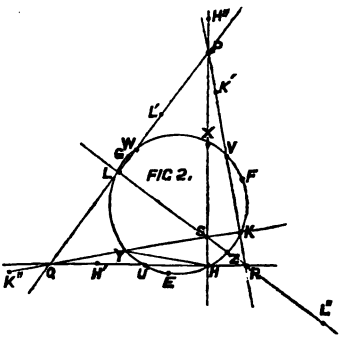

Cor. 2.—If HU and HX (Fig. 2) be any pair of rectangular tangents intersecting at H on the inscribed circle, and QS the segment they intercept on any third tangent KY, the segment QS is always bisected by the circle at the centre Y of the tangent; and conversely, if the circle bisect the segment of a line intercepted by any pair of rectangular tangents to the curve, the line is a tangent, and the point of bisection its centre.

For since, by (II.), when KY is a tangent, arc EK = 2 arc EY, and arc EH = 2 arc EU, therefore arc HK = 2 arc UY, and consequently angle HYS = 2 angle YHQ; and therefore the first part follows, since QHS is a right angle. Conversely, when QS is bisected at Y, then, QHS being a right angle, HYK = 2UHY, and therefore arc EK = 2 arc EY; and therefore, by (II.), the remaining part follows.

N.B.—If, while the pair of rectangular tangents HU and HX are supposed to remain fixed, the third tangent KY be conceived to vary, then, since in every position the triangle HYQ is isosceles, and the two bisectors of its vertical angle Y consequently parallel in every position to the fixed pair of tangents HU and HX, it follows from the above that *if while the vertex* Y *of a variable angle* HYQ *moves on a fixed circle, one of its sides* HY *turns round a fixed point* H *on the circle, and its bisectors are constantly parallel to two fixed rectangular chords* HU *and* HX *passing through the point, the other side* YK *will envelop a tricusped hypocycloid which touches the circle at the three points of trisection* E, F, G *of the four arcs intercepted by the two chords.*

Cor. 3.—If P and Q, R and S, (Fig. 2.) be the two pairs of opposite intersections of any two different pairs of rectangular tangents HU and HX, KV and KY, their two lines of connection PQ and RS determine a third pair of rectangular tangents LW and LZ. For, the two pairs of lines HU and HX, KV and KY, being, by hypothesis, rectangular, the four points P, Q, R, S determine a tetrastigm whose three pairs of opposite connectors, QR and PS, RP and QS, PQ and RS, are rectangular; and since the inscribed circle, which, by Cor. 1, passes through the two points of intersection H and K, passes also, by Cor. 2, through the four points of bisection U and X, V and Y of the two pairs of connectors QR and PS, RP and QS, it is consequently the nine-point circle of the tetrastigm, and therefore passes through the point of intersection L, and through the two points of bisection W and Z of the remaining pair of connectors PQ and RS, which connectors are consequently, by Cor. 2, tangents to the figure.

N.B.—Of the four points P, Q, R, S, through each of which three concurrent tangents pass, and each of which is the polar centre of the triangle determined by the remaining three, each, taken by itself, irrespectively of the others, being of course entirely arbitrary, it follows consequently from the above, that, *when three tangents to a tricusped hypocycloid are concurrent, the three rectangular tangents determine a triangle, of which the point of concurrence is the polar centre, and of which the inscribed circle of the figure is the nine-point circle.*

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Cor. 4.—If H and H″, K′ and K″, L′ and L″, (Fig. 2.) be the six points of contact of the three pairs of tangents HU and HX, KV and KY, LW and LZ, then, since, by (I.), the six segments HH′ and HH″, KK′ and KK″, LL′ and LL″, are concentric with the six QR and PS, RP and QS, PQ and RS, at the six points U and X, V and Y, W and Z respectively, and since, in the four triangles PQR, QSR, RSP, PSQ, the three perpendiculars to the sides at the three points H, K, L are concurrent, therefore in the same (*Mod. Geom.* Art 133, Ex. 6), the four triads of perpendiculars to the sides at the four triads of points H′, K′, L′; L″, H′, K″; H″, K′, L″; K″, L′, H″, are concurrent, and the four points of concurrence connect evidently with the four corresponding points S, P, Q, R respectively, by four lines passing through the centre of the common nine-point circle, which point divides them all in the common ratio of 3 : 1. Hence, *when three tangents to a tricusped hypocycloid are concurrent, the three normals at the points of contact of the three rectangular tangents are also concurrent, and the two points of concurrence connect through the centre of the inscribed circle of the figure from which point they are distant in opposite directions in the constant ratio of* 3 : 1.

N.B.—Fig. 1. supplies an example of a particular case of the above, the three tangents ZP, ZQ, ZK being concurrent at Z, and the normals QM, PL, RN, corresponding to the three perpendicular tangents ZQ, ZP, PQ, being also concurrent at N; also, the line NZ passing through O, and being there divided so that ON : OZ = 3 : 1.

Cor. 5.—In the same four triangles PQR, QSR, RSP, PSQ, (Fig. 2.), the four triads of connectors PH, QK, RL; QL, RK, SH; RH, PL, SK; PK, QH, SL being concurrent, therefore in the same (*Mod. Geom.* Art 137, Ex. 11.) the four triads of connectors PH′, QK′, RL′; QL″, RK″, SH′; RH″, PL″, SK′; PK″, QH″, SL′ are concurrent, and therefore, *when three tangents to a tricusped hypocycloid are concurrent, the three rectangular tangents determine a triangle whose vertices connect concurrently with the points of contact of the opposite sides.*

N.B.—From the converse of this latter property combined with that of the preceding corollary it follows immediately that, *if a conic have triple contact with a tricusped hypocycloid, the three normals at the three points of contact are concurrent; and the line connecting their point of concurrence with the polar centre of the triangle determined by the three tangents passes through the centre of the figure, and is there divided in the constant ratio of* 3 : 1.

Cor. 6.—As every equilateral hyperbola which passes through any three of the four points P, Q, R, S, (Fig. 2.) passes, as is well known, through the fourth, and has its centre on the nine-point circle of the tetrastigm they determine; and as the pair of rectangular tangents to the figure, which intersect at that or any other point on that circle, intercept (by Cor. 2.) on the six lines of that tetrastigm six segments concentric with the six connectors QR and PS, RP and QS, PQ and RS; it follows consequently that those tangents are the asymptotes of the hyperbola having its centre at their point of intersection, and therefore that, *when four points are such that each is the polar centre of the triangle determined by the remaining three, the asymptotes of every equilateral hyperbola which passes through any three of them (and therefore through the fourth) are tangents to the tricusped hypocycloid which touches the three pairs of opposite connectors of the tetrastigm they determine.*

Cor. 7.—If A, B, C (Fig. 3) be the three vertices of any triangle; D its polar centre; P any point on its circumscribing circle; P′ the diametrically

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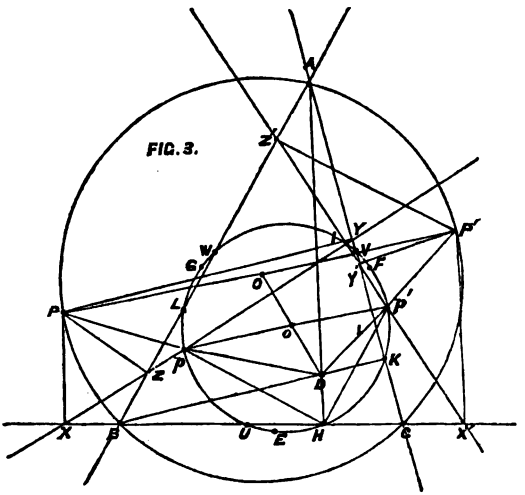
opposite point; X, Y, Z the feet of the three perpendiculars from P, and X′, Y′, Z′ those of the three from P′ on its three sides BC, CA, AB, respectively; the two lines of collinearity XYZ and X′Y′Z′ intersect at right angles on its nine-point circle, and are tangents to the tricusped hypocycloid which touches its three sides and perpendiculars, and consequently also touches that circle at the three points of trisection E, F, G, common to the six pairs of arcs into which it is divided by the six lines.

FIG. 3.



For if *pop′* be the diameter of the nine-point circle parallel to the diameter POP′ of the circumscribing circle; and H the foot of any one of the three perpendiculars AH, BK, CL of the triangle; then, the polar centre D being the external centre of similitude of the two circles, the two lines DP and DP′ pass through and are bisected at *p* and *p′*, which therefore (see Solution to Quest. 1649) are points on the two lines XYZ and X′Y′Z′; hence the two triangles X*p*H and X′*p′*H are isosceles, and therefore (Cor. 2, Note) the two lines XYZ and X′Y′Z are tangents to the hypocycloid in question; and as their centres *p* and *p′* are diametrically opposite points of its inscribed circle, they therefore (Cor. 1) intersect at right angles on that circle.

N.B.—From the converse of this latter property, combined with that of the preceding Corollary, it follows, indirectly, that *the two triads of perpendiculars to the three sides of a triangle at the two triads of points at which they are met by the asymptotes of any equilateral hyperbola passing through its three vertices concur to two diametrically opposite points on its circumscribing circle*—a property which may be easily proved directly.

1710. (Proposed by Professor CAYLEY.)—Trace the curve $y^4 - 2y^2zx - x^4 = 0$, where the coordinates are such that $x + y + z = 0$ is the line *infinity*.

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Solution by the PROPOSER.

We have $x = \frac{y^4 - z^4}{2y^2 z}$; or writing $y = \theta z$, then $x = \frac{\theta^4 - 1}{2\theta^2} z$, that is

$$x : y : z = \theta^4 - 1 : 2\theta^3 : 2\theta^2.$$

Hence, we see that y, z are indefinitely small in comparison of x,

if $\theta = \infty$, and then $x : y : z = \theta^4 : 2\theta^3 : 2\theta^2$, that is $y^2 = 2zx$;

or, if $\theta = 0$, and then $x : y : z = -1 : 2\theta^3 : 2\theta^2$, that is $z^3 = -2y^2 x$;

so that in the neighbourhood of the point $(y = 0, z = 0)$ there are two branches coinciding with the parabola $y^2 = 2zx$ and with the semicubical parabola $z^3 = -2y^2 xz$, respectively.

For the points at infinity we have $x + y + z = 0$, that is $\theta^4 + 2\theta^3 + 2\theta^2 - 1 = (\theta + 1)(\theta^3 + \theta^2 + \theta - 1) = 0$; and observing that the equation $\theta^3 + \theta^2 + \theta - 1 = 0$ has one real root, say $\theta = k$, if k be the real root of the equation $k^3 + k^2 + k - 1 = 0$ ($k = .505$ nearly),—there are two real points at infinity, viz., corresponding to $\theta = -1$, we have the point $(0, -1, 1)$, and corresponding to $\theta = k$ the point $(-1 - k, k, 1)$.

The equation of the tangent at a point (α, β, γ) is

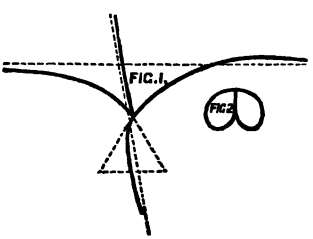
$$x(-\beta^2\gamma) + y(2\beta^3 - 2\alpha\beta\gamma) + z(-\alpha\beta^2 - 2\gamma^3) = 0,$$

and hence writing $(\alpha, \beta, \gamma) = (0, -1, 1)$ we have the asymptote $x + 2y + 2z = 0$: to find where this meets the curve, we have $\theta^4 + 4\theta^3 + 4\theta^2 - 1 = 0$, that is $(\theta + 1)^2(\theta^2 + 2\theta - 1) = 0$, or at the points of intersection $\theta^2 + 2\theta - 1 = 0$, that is $\theta = -1 \pm \sqrt{2}$, or there are two real points of intersection.

Again writing $(\alpha, \beta, \gamma) = (-1 - k, k, 1)$ we find the asymptote $k^2 x - 2y + (k + 1)z = 0$: to find where this meets the curve, we have $k^2(\theta^4 - 1) - 4k\theta^3 + (2k + 2)\theta^2 = 0$, that is $k^2\theta^4 - 4k\theta^3 + (2k + 2)\theta^2 - k^2 = (\theta - k)^2\{k^2\theta^2 - 2(k^2 + k + 1)\theta - 1\} = 0$; or for the points of intersection $k^2\theta^2 - 2(k^2 + k + 1)\theta - 1 = 0$, an equation in θ with two real roots, that is the points of intersection are real.

It is now easy to lay down the curve; viz., if, to fix the ideas, the fundamental triangle is taken to be equilateral, and the coordinates x, y, z are considered to be positive for points *within* the triangle, then the curve is as shown in the annexed figure (Fig. 1.)

It may be remarked that the curve is met by every real line in two real points at least, and consequently that it is not the projection of any finite curve whatever. By a modification of the constants of the equation, we might obtain curves which are finite, such as the curve in Fig. 2; or curves with two or four infinite branches, which are the projections of such a finite curve.



1717. (Proposed by W. S. BURNSIDE, B.A.)—Show how the locus of the foci of conics, subject to four conditions, may be found in the following

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cases :—(1.) When each conic of the system touches three lines, and satisfies any fourth condition. (2.) When each conic passes through three fixed points, and satisfies any fourth condition. (3.) When a given triangle is self-conjugate with regard to each conic of the system, which also satisfies any fourth condition.

Solution by T. A. HIRST, F.R.S.

If ν denote the number of conics, subject to four conditions, which touch an arbitrary right line, then according to Chasles' theory, the locus of their foci is a curve of the order 3ν, which has multiple points of the order ν at each of the circular points at infinity. This important theorem is easily demonstrated on replacing, as usual, the two circular points by any two points whatever P and Q, which have no special relation to the system of conics under consideration. We have then merely to examine the locus of the intersections of tangents from P and Q to the several conics of the system, or rather to find the number of such intersections which lie on an arbitrary line T, passing through P. Now, by hypothesis, there are ν conics which touch T, and to every such conic two tangents can be drawn from Q, so that, exclusive of the point P itself, every transversal T contains 2ν of the intersections in question. But P itself obviously represents ν such intersections, since there are ν conics of the system which touch the line PQ.

The theorem, therefore, is established, and may be at once applied to the three special cases alluded to in the question. I remark that in (1), the *characteristic* ν is obviously the number of conics inscribed in the given triangle ABC, which, besides touching an arbitrary line, satisfy the given fourth condition. The locus of the foci cuts each of the given lines, say BC, in 3ν points. But the conic inscribed to ABC and having one of its foci on BC obviously degenerates to the rectilinear connector of its two foci, of which latter one must necessarily coincide with an angle of the triangle ABC. Hence, and from the fact that A, B, and C must be similarly related to the locus under consideration, we infer that the latter, besides being of the order 3ν and circular in the order ν, has also multiple points of the latter order at the intersections of the three given lines.

In (2) and (3), the fourth conditions being alike, the characteristics ν will be equal to one another. For, as far as the characteristics are concerned, the condition that each conic of a system shall cut a given segment harmonically is equivalent to the condition that each shall pass through a given point. A demonstration of this statement of Chasles is proposed in Question 1732.

II. *Solution by the* PROPOSER.

1. Take as triangle of reference that formed by the three tangents; then the equation of each conic is of the form $(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0$, and the tangential equation is $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$; hence, putting Ω for $\alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma \cos A - 2\gamma\alpha \cos B - 2\alpha\beta \cos C$, and Σ for $2l\beta\gamma + 2m\gamma\alpha + 2n\alpha\beta$, if λ be so determined that $\Sigma + \lambda\Omega \equiv (\alpha x + \beta y + \gamma z)(\alpha x_1 + \beta y_1 + \gamma z_1)$, the points (x, y, z), (x_1, y_1, z_1) are foci (Salmon's *Conics*, chap. xviii.).

Comparing both sides of this identity, we have

$$xx_1 = \lambda \ldots\ldots\ldots (1), \qquad yz_1 + y_1z = 2(l - \lambda \cos A) \ldots\ldots (4),$$
$$yy_1 = \lambda \ldots\ldots\ldots (2), \qquad zx_1 + z_1x = 2(m - \lambda \cos B) \ldots\ldots (5),$$
$$zz_1 = \lambda \ldots\ldots\ldots (3), \qquad xy_1 + x_1y = 2(n - \lambda \cos C) \ldots\ldots (6).$$

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From (2), (3), (4), $2lyz = y^2 + 2yz \sin A + z^2 = P$, say;

„ (3), (5), (1), $2mzx = z^2 + 2zx \cos B + x^2 = Q$, say;

„ (1), (2), (6), $2nxy = x^2 + 2xy \cos C + y^2 = R$, say;

hence we have $l : m : n = Px : Qy : Rz$.

And as any fourth condition gives a homogeneous equation in (l, m, n), we may easily find the locus of the foci in this case by substituting for these quantities (Px, Qy, Rz).

2. Taking as triangle of reference that formed by the three points, the equation of each conic of the system is of the form $lyz + mzx + nxy = 0$, and the tangential equation is $l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0$ (or Σ); and if $\Sigma - \lambda\Omega \equiv (ax + \beta y + \gamma z)(a_1x + \beta_1 y + \gamma z_1)$, as before we have

$$\begin{aligned} xx_1 &= l^2 - \lambda \ldots\ldots (1), & yz_1 + y_1z &= 2(\lambda \cos A - mn) \ldots. (4), \\ yy_1 &= m^2 - \lambda \ldots\ldots (2), & zx_1 + z_1x &= 2(\lambda \cos B - nl) \ldots. (5), \\ zz_1 &= n^2 - \lambda \ldots\ldots (3), & xy_1 + x_1y &= 2(\lambda \cos C - lm) \ldots. (6). \end{aligned}$$

From (2), (3), (4); (3), (5), (1); (1), (2), (6); we have

$$(ny + mz)^2 = \lambda P, \quad (lz + nx)^2 = \lambda Q, \quad (mx + ly)^2 = \lambda R;$$

hence

$$l : m : n = x(-xP^{\frac{1}{2}} + yQ^{\frac{1}{2}} + zR^{\frac{1}{2}}) : y(xP^{\frac{1}{2}} - yQ^{\frac{1}{2}} + zR^{\frac{1}{2}}) : z(xP^{\frac{1}{2}} + yQ^{\frac{1}{2}} - zR^{\frac{1}{2}}).$$

And as any fourth condition gives a homogeneous equation in (l, m, n), we can eliminate $l : m : n$ by the above proportion, and determine the locus of the foci in this case.

3. Taking as triangle of reference the self-conjugate triangle, the equation of each conic of this system is of the form $ax^2 + by^2 + cz^2 = 0$, and the tangential equation is $bc\alpha^2 + ca\beta^2 + ab\gamma^2 = 0$ or $A\alpha^2 + B\beta^2 + C\gamma^2 \equiv \Sigma$; and $\Sigma - \lambda\Omega \equiv (ax + \beta y + \gamma z)(a_1x + \beta_1 y + \gamma z_1)$ gives

$$\begin{aligned} xx_1 &= A - \lambda \ldots\ldots (1), & yz_1 + y_1z &= 2\lambda \cos A \ldots\ldots (4), \\ yy_1 &= B - \lambda \ldots\ldots (2), & zx_1 + z_1x &= 2\lambda \cos B \ldots\ldots (5), \\ zz_1 &= C - \lambda \ldots\ldots (3), & xy_1 + x_1y &= 2\lambda \cos C \ldots\ldots (6). \end{aligned}$$

From (2), (3), (4); (3), (1), (5); (1), (2), (6); we have

$$Cy^2 + Bz^2 = \lambda P, \quad Az^2 + Cx^2 = \lambda Q, \quad Bx^2 + Ay^2 = \lambda R;$$

hence

$$A : B : C = x^2(-x^2P + y^2Q + z^2R) : y^2(x^2P - y^2Q + z^2R) : z^2(x^2P + y^2Q - z^2R)$$

or $$bc : ca : ab = xyz + x(-x \cos A + y \cos B + z \cos C)$$

$$: xyz + y(x \cos A - y \cos B + z \cos C) : xyz + z(x \cos A + y \cos B - z \cos C),$$

substituting for A, B, C and P, Q, R their values. We can thus find $a : b : c$ easily, since $bc : ca : ab = a^{-1} : b^{-1} : c^{-1}$.

And as any fourth condition gives a homogeneous equation in (a, b, c), we can easily eliminate these quantities by the above proportion, and find the locus of the foci in this case.

1749. (Proposed by the Rev. W. ROBERTS, M.A.)—AB is a diameter of a circle, O its centre, and P a point in OA, such that $OP = \frac{1}{3}OA$. Through P

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any straight line QPR is drawn, meeting the circumference in Q, R; and the straight line passing through Q, O is produced to meet that through B, R in S.

If $\angle OSB = \psi$, and $\angle OBS = \phi$, prove that $\frac{1-\sin\psi}{1+\sin\psi} = \left(\frac{1-\sin\phi}{1+\sin\phi}\right)^3$.

Solution by DR. BOOTH, F.R.S.

The following solution of this question will illustrate a new method of analysis, which I have called *Parabolic Trigonometry*, of which an outline was given in a paper read before the British Association at Cheltenham in 1856, and published among the *Reports* of that year. A reference to the same theory will be found in the *Philosophical Transactions* for 1852, p. 386.

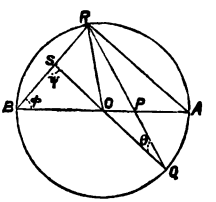
The formula in the Question may be reduced to

$$\sec\psi + \tan\psi = (\sec\phi + \tan\phi)^3;$$

and this latter expression, as I have shown in the paper above referred to, may be put in the form

$$(\sec\phi + \tan\phi)^3 = \sec(\phi \perp \phi \perp \phi) + \tan(\phi \perp \phi \perp \phi),$$

in which $\perp$ and $\top$ denote a relation which I have termed parabolic or logarithmic *plus* and *minus*.



Now, drawing the annexed figure according to the construction given, let $PQ = a$, $OQ = r$, and $\angle OQP = \theta$; then we have

$$2\theta = \psi - \phi, \quad 3a\sin\theta = r\sin(\psi + \phi),$$
$$9a^2 = 10r^2 + 6r^2\cos(\psi + \phi);$$

hence, eliminating θ, a, r, we have

$$2(\sec\phi\sec\psi - \tan\psi\tan\phi)^2$$
$$-(\sec\phi\sec\psi + \tan\phi\tan\psi) = 1.$$

Subtracting $(\sec^2\psi - \tan^2\psi)(\sec^2\phi - \tan^2\phi) = 1$ from the preceding equation, we have

$$(\sec^2\psi + \tan^2\psi)(\sec^2\phi + \tan^2\phi) - 4\sec\psi\tan\psi\sec\phi\tan\phi$$
$$= \sec\phi\sec\psi + \tan\psi\tan\phi.$$

Now, in the paper referred to, it is shown that

$$\sec^2\psi + \tan^2\psi = \sec(\psi \perp \psi), \text{ &c.}; \quad 2\sec\psi\tan\psi = \tan(\psi \perp \psi), \text{ &c.};$$
$$\sec\psi\sec\phi + \tan\phi\tan\psi = \sec(\psi \perp \phi), \text{ &c.};$$

hence, substituting these values in the preceding equation, it becomes

$$\sec(\psi \perp \psi)\sec(\phi \perp \phi) - \tan(\psi \perp \psi)\tan(\phi \perp \phi) = \sec(\psi \perp \phi).$$

But this formula may be written

$$\sec(\psi \perp \psi \top \phi \top \phi) = \sec(\psi \perp \phi), \quad \text{or} \quad \psi \perp \psi \top \phi \top \phi = \psi \perp \phi;$$

or, transposing and changing $\top$ into $\perp$,

$$\psi = \phi \perp \phi \perp \phi, \quad \text{or} \quad \sec\psi + \tan\psi = (\sec\phi + \tan\phi)^3.$$

The following general relation exists in Parabolic Trigonometry; viz.,

$$\sec(\phi \perp \phi \perp \ldots \text{to } n \text{ angles}) + \tan(\phi \perp \phi \perp \ldots \text{to } n \text{ angles})$$
$$= (\sec\phi + \tan\phi)^n,$$

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just as in Circular Trigonometry, if $i = \sqrt{(-1)}$, we have

$$\cos n\phi + i \sin n\phi = (\cos \phi + i \sin \phi)^n.$$

Thus De Moivre's celebrated theorem on the relation of circular arcs has its dual in the real relations of parabolic arcs.

The curve of which $r = a(\sec \phi + \tan \phi)$ is the polar equation, I have called the *Logocyclic Curve*. One of its most important properties is that, if we represent all numbers from 0 to ∞ in succession by the radii of this curve, the conjugate parabola to it will represent by its arcs the logarithms of those numbers.

[NOTE.—The theorem may be otherwise proved as follows :—

$$2 = \frac{PB}{PA} = \frac{PB}{PR} \cdot \frac{PR}{PA} = \frac{\sin \frac{1}{2}(\phi + \psi)}{\sin \phi} \cdot \frac{\cos \phi}{\cos \frac{1}{2}(\phi + \psi)};$$

therefore

$$\tan \tfrac{1}{2}(\phi + \psi) = 2 \tan \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha);$$

whence we obtain, successively,

$$\tan \tfrac{1}{2}\psi = \frac{\tan^3 \frac{1}{2}\phi + 3 \tan \frac{1}{2}\phi}{3 \tan^2 \frac{1}{2}\phi + 1}; \quad \frac{1 - \tan \frac{1}{2}\psi}{1 + \tan \frac{1}{2}\psi} = \left(\frac{1 - \tan \frac{1}{2}\phi}{1 + \tan \frac{1}{2}\phi}\right)^3;$$

$$\frac{1 - \sin \psi}{1 + \sin \psi} = \left(\frac{1 - \sin \phi}{1 + \sin \phi}\right)^3.$$

Moreover, putting $OS = \rho$, $\angle AOS = \chi$, we have

$$\frac{r}{\rho} = \frac{\sin \psi}{\sin \phi} = \frac{\sin (\chi - \phi)}{\sin \phi} = \sin \chi \cot \phi - \cos \chi \quad \ldots\ldots\ldots (\beta).$$

But (α) is equivalent to $\cot \phi = 2 \cot \frac{1}{2}\chi$, hence (β) becomes $\rho(1 + \frac{1}{2} \cos \chi) = \frac{1}{2}r$, which shows that the *locus of* S is an ellipse whose eccentricity is $\frac{1}{2}$, major axis BP, and focus O.—EDITOR.]

ON RADIAL CURVES.

By R. TUCKER, M.A.

66. In continuation of previous Articles on Radials, we proceed to prove the following equation which we have stated elsewhere. Adopting the notation previously used, and denoting by p' the perpendicular from the Radial pole on the tangent at any point, we have

$$p' = \rho \sin \phi' = -\left(\frac{ds}{d\phi}\right)^2 \div \frac{ds''}{d\phi};$$

hence

$$\frac{dp'}{d\phi} = \left\{-2\frac{ds}{d\phi} \cdot \frac{d^2s}{d\phi^2} \cdot \frac{ds''}{d\phi} + \frac{d^2s''}{d\phi^2} \cdot \left(\frac{ds}{d\phi}\right)^2\right\} \div \left(\frac{ds''}{d\phi}\right)^2;$$

and

$$R_1 = \frac{d^2s''}{d\phi_1^2} = \frac{d^2s''}{d\phi^2} \cdot \left(\frac{d\phi}{d\phi_1}\right)^2; \quad \therefore \frac{d^2s''}{d\phi^2} = R_1 \cdot \frac{\rho^2 + R^2}{\rho_1^2};$$

whence

$$\frac{dp'}{d\phi} = -\frac{\rho}{\rho_1^2}\left\{\frac{2R\rho_1^2 - \rho R_1 \sqrt{(\rho^2 + R^2)}}{\sqrt{(\rho^2 + R^2)}}\right\};$$

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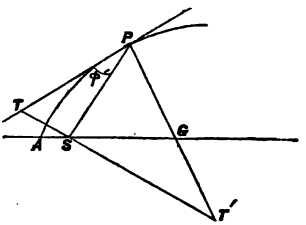
but $\rho_1 = \rho \dfrac{d\rho}{dp'} = R\rho_1^2 \surd(\rho^2 + R^2) \div \left\{2R\rho_1^2 - \rho R_1 \surd(\rho^2 + R^2)\right\}$;

hence $2R\rho_1^2 - R\rho_1 \surd(\rho^2 + R^2) = \rho R_1 \surd(\rho^2 + R^2)$

or, by (ξ), $R\rho_1 \left\{2 \dfrac{d\phi}{d\phi_1} - 1\right\} = \rho R_1$ } ..... (τ).



67. Let PT be the tangent at any point P of the radial AP (pole S), and TST′ a perpendicular to SP meeting the normal PG in T′; then, by (δ) and (ε), we have polar subnormal = ST′ = $\rho \cot \phi'$ = radius of curvature at corresponding point of evolute of primitive; and the direction of ST′ is clearly parallel to the direction of the same radius of curvature; hence we readily see that the locus of T′ is the radial of the evolute of the primitive. This furnishes an easy method for finding the radial of the evolute of a curve when the radial of the primitive is known.

68. Now, since if a curve and its evolute are similar their radials must be similar, we see that the problem of Art. 24 is identical with that of finding when the locus of T′ is similar to that of P. We thus easily arrive at the functional equation and result of Art. 24.

69. If we suppose SP, ST′ corresponding lines, we have $\dfrac{ST'}{SP}$ = a constant = $\cot \phi'$, which is the property of an equiangular spiral. (See above, Art. 25.)

70. If T″, T‴, ... T_n be a series of points (lying alternately on SP, ST′) obtained as in Art. 67, we see that the locus of T_n is the radial of the nth evolute of the primitive of AP.

71. When the locus of T′ is the inverse of that of P, and SP, ST′ corresponding lines, since in this case

$$k^2 = SP \cdot ST' = r'^2 \cot \phi' = r' \frac{dr'}{d\theta'},$$

we have $r'^2 = 2k^2\theta'$ for the locus of P.

72. We will now show that a somewhat more general solution may be found for equation (A) in Art. 24. The case we propose to discuss is that of $n = 1$. If ϵ be the base of Napierian logarithms, our equations will then be

$$\frac{2x}{c} = \int 2\epsilon^{m\theta} \cos^2 \theta \, d\theta = \frac{\epsilon^{m\theta}}{m} \left\{1 + \frac{m^2 \cos 2\theta + 2m \sin 2\theta}{m^2 + 4}\right\} + c_1,$$

$$\frac{2y}{c} = \int \epsilon^{m\theta} \sin 2\theta \, d\theta = \epsilon^{m\theta} \left\{\frac{m \sin 2\theta - 2 \cos 2\theta}{m^2 + 4}\right\} + c_2;$$

and if now we suppose x and y to vanish with θ, we have

$$c_1 = -\frac{2(m^2 + 2)}{m(m^2 + 4)}, \quad c_2 = \frac{2}{m^2 + 4};$$

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hence, re-arranging, we get

$$\epsilon^{m\theta}(m^2\cos 2\theta + 2m\sin 2\theta + m^2 + 4) = \frac{2x}{c}m(m^2+4) + 2(m^2+2) = X,$$

$$\epsilon^{m\theta}(m^2\sin 2\theta - 2m\cos 2\theta) = \frac{2y}{c}m(m^2+4) - 2m = Y;$$

squaring and adding, we have

$$X^2 + Y^2 = 2(m^2+4)\epsilon^{2m\theta}(m^2 + 2 + m^2\cos 2\theta + 2m\sin 2\theta),$$

and $$X = \epsilon^{m\theta}(m^2\cos 2\theta + 2m\sin 2\theta + m^2 + 4);$$

whence we obtain

$$2\epsilon^{2m\theta} - X\epsilon^{m\theta} + \frac{X^2+Y^2}{2(m^2+4)} = 0,$$

or $$\epsilon^{m\theta} = \left\{X\sqrt{(m^2+4)} \pm \sqrt{(m^2X^2 - 4Y^2)}\right\} \div 4\sqrt{(m^2+4)} = t^m;$$

therefore $$\theta = \log t,$$

and $$Y = t^m \cdot \left\{m^2\sin(\log t^2) - 2m\cos(\log t^2)\right\},$$

the equation to a class of curves whose evolutes are similar to themselves. If we make $m = 0$, we get (as we ought) the cycloid as a particular curve of the class.

73. We next proceed to find what curves have for radials the conic sections, the radial pole being the focus of the conic.

We take the equation (e being the eccentricity)

$$r = \frac{c}{1 + e\cos\theta}, \quad \text{hence} \quad \frac{dp}{(1+p^2)\left\{\sqrt{(1+p^2)} + ep\right\}} = \frac{dx}{c} = \frac{\cos z\, dz}{1 + e\sin z}$$

(if $p = \tan z$); integrating, $\frac{ex}{c} = \log(1 + e\sin z)$,

or $$\epsilon^{\frac{ex}{c}} = 1 + e\frac{p}{\sqrt{(1+p^2)}}, \quad \text{and} \quad \frac{dy}{dx} = p = \frac{\epsilon^{\frac{ex}{c}} - 1}{\sqrt{\left\{e^2 - (\epsilon^{\frac{ex}{c}} - 1)^2\right\}}};$$

whence, assuming $\epsilon^{\frac{ex}{c}} - 1 = e\cos\phi$, we get

$$\frac{e}{c}dy = -d\phi + \frac{d\phi}{1 + e\cos\phi};$$

therefore $$ey = \frac{c}{\sqrt{(e^2-1)}}\log\frac{\sqrt{(e-1)}\tan\frac{1}{2}\phi + \sqrt{(e+1)}}{\sqrt{(e-1)}\tan\frac{1}{2}\phi - \sqrt{(e+1)}} - c\phi \ldots\ldots (e>1),$$

also $$ey = \frac{2c}{\sqrt{(1-e^2)}}\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\tfrac{1}{2}\phi - c\phi \quad \ldots\ldots\ldots (e<1),$$

and $$y = c\tan\tfrac{1}{2}\phi - c\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (e=1),$$

where ϕ has the value above assigned.

74. Next, when the radial pole is the centre of the conic. We must discuss the curves separately, and first we take the ellipse, whose equation is

$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta};$$

$$\therefore \frac{dx}{b} = \frac{dp}{(1+p^2)\sqrt{\left\{1 + p^2(1-e^2)\right\}}} = \frac{\cos z\, dz}{\sqrt{(1 - e^2\sin^2 z)}} \text{ (if } p = \tan z);$$

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$$\therefore \frac{x}{b} = \sin^{-1}(e \sin z), \quad \text{and} \quad \sin\frac{x}{b} = e \sin z = e\frac{p}{\sqrt{(1+p^2)}};$$

therefore
$$\frac{dy}{dx} = p = \frac{-bd\left(\cos\frac{x}{b}\right)}{\sqrt{\left\{\cos^2\frac{x}{b} - (1-e^2)\right\}}};$$

whence
$$\epsilon^{-\frac{y}{b}} = \cos\frac{x}{b} + \sqrt{\left\{\cos^2\frac{x}{b} - (1-e^2)\right\}};$$

and
$$\frac{b^2}{a^2}\epsilon^{\frac{y}{b}} = \cos\frac{x}{b} - \sqrt{\left\{\cos^2\frac{x}{b} - (1-e^2)\right\}};$$

therefore
$$\frac{b^2}{a^2}\epsilon^{\frac{y}{b}} + \epsilon^{-\frac{y}{b}} = 2\cos\frac{x}{b}$$

is the equation to the primitive of the ellipse referred to its centre.

75. Similarly for the hyperbola we have

$$r^2 = \frac{b^2}{e^2\cos^2\theta - 1}, \quad \text{and} \quad \frac{dx}{b} = \frac{\cos z\, dz}{\sqrt{(e^2\sin^2 z - 1)}};$$

therefore
$$\frac{ex}{b} = \log\left\{e \sin z + \sqrt{(e^2\sin^2 z - 1)}\right\};$$

hence
$$\epsilon^{\frac{ex}{b}} + \epsilon^{-\frac{ex}{b}} = 2e \sin z = 2e\frac{p}{\sqrt{(1+p^2)}};$$

therefore, if $\epsilon^{\frac{ex}{b}} + \epsilon^{-\frac{ex}{b}} = z'$, we have

$$dy = \frac{bz'\,dz'}{e\sqrt{(4e^2 - z'^2)}\sqrt{(z'^2-4)}} = \frac{bd\,(z')^2}{2e\sqrt{\left\{4(e^2+1)z'^2 - z'^4 - 16e^2\right\}}},$$

and
$$\frac{2ey}{b} = \sin^{-1}\left\{\frac{2z'^2 - 4(e^2+1)}{4(e^2-1)}\right\},$$

or
$$2(e^2-1)\sin\frac{2ey}{b} = \epsilon^{\frac{2ex}{b}} + \epsilon^{-\frac{2ex}{b}} - 2e^2,$$

which reduces, in the case of the rectangular hyperbola, to

$$2\cos\left(\frac{\pi}{4} - \frac{y\sqrt{2}}{b}\right) = \epsilon^{\frac{x\sqrt{2}}{b}} - \epsilon^{-\frac{x\sqrt{2}}{b}}.$$

76. From equation (ι) we have

$$\frac{ds''}{dx} = \frac{(1+p^2)^{\frac{1}{2}}\left\{q^4 + [3pq^2 - r(1+p^2)]^2\right\}^{\frac{1}{2}}}{q^4} \quad \ldots\ldots\ldots\ldots\ldots (\upsilon),$$

an equation connecting the arc of a radial with the abscissa of the corresponding point on the primitive.

77. We may apply (υ) to the case considered in Art. 55. Since the curve and radial are similar, if s, s'' are corresponding arcs, we have,

since
$$\sqrt{(1+p^2)} = \frac{ds}{dx}, \quad k^2q^4 = q^4 + [3pq^2 - r(1+p^2)]^2;$$

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hence $$bq^2 = 3pq^2 - r(1+p^2);$$

therefore $$\frac{r}{q} = \frac{(3p-b)q}{1+p^2}, \quad \text{and} \quad b\tan^{-1}p = \log\frac{(1+p^2)^{\frac{3}{2}}}{q};$$

therefore $r' = ce^{-b\theta'}$ is the equation to the radial, or the curve required, by Art. 17, is the equiangular spiral.

78. From our previous equations we can readily find the locus of a point whose distance from a curve equals the radius of curvature at the corresponding point of that curve. There are two curves fulfilling the conditions, the evolute and the locus we are about to indicate.

Let X′, Y′ be the coordinates of the point; then

$$X' = 2x - \alpha = x + x', \quad Y' = 2y - \beta = y - y';$$

therefore $$dY' = 2dy - d\beta = dy - dy', \quad dX' = 2dx - d\alpha = dx + dx';$$

and $$(dS')^2 = 4(ds)^2 + (ds')^2 = 3(ds)^2 + (ds'')^2;$$

$$\frac{dY'}{dX'} = \frac{x'y' - p'(2x'^2 + y'^2)}{x'^2 + 2y'^2 - p'x'y'}, \text{ by } (\theta), \text{ or } = \frac{-pq^2 + r(1+p^2)}{q^2(2+3p^2) - rp(1+p^2)} \text{ by } (\iota),$$

$$\frac{d^2Y'}{dX'^2} = \frac{q^3\left\{-2q^4 - 2pq^2r(1+p^2) + (1+p^2)^2(2qs - 3r^2)\right\}}{\left\{q^2(2+3p^2) - rp(1+p^2)\right\}^2};$$

$$P' = \frac{(1+p^2)^{\frac{3}{2}}}{q^3} \cdot \frac{\left\{4q^4 + [3pq^2 - r(1+p^2)]^2\right\}^{\frac{3}{2}}}{(1+p^2)^2(2qs - 3r^2) - 2pq^2r(1+p^2) - 2q^4}.$$

79. By elementary geometry we have

$$X'^2 + Y'^2 + \alpha^2 + \beta^2 = 2(y^2 + x^2) + 2(y'^2 + x'^2).$$

80. From our equations (Art. 78) we can show that the circle is the only curve for which this locus is parallel to original curve; also that for the catenary the locus is the axis of X, as we can infer from a known property of the curve.

81. Since $\frac{dy}{dx} = \cot\theta'$ $\therefore \frac{d^2y}{dx^2} = -\operatorname{cosec}^2\theta'\frac{d\theta'}{dx} = \frac{1}{r'\sin^3\theta'}$ by (β);

hence $$\frac{dx}{d\theta'} = -r'\sin\theta' = -Y, \text{ and } \frac{dy}{d\theta'} = -r'\cos\theta' = -X;$$

therefore $$\frac{d^3y}{dx^3} = \frac{\sin\theta'\frac{dr'}{d\theta'} + 3r'\cos\theta'}{r'^3\sin^5\theta'},$$

and similarly for higher differential coefficients.

82. Let v, v', v'' be the velocities of particles at corresponding points of the primitive, its evolute, and its radial, then, employing the usual notation, we have

$$v''^2 = \frac{h''^2}{p'^2} = \frac{h''^2(ds'')^2}{r'^2(ds)^2} = \frac{h''^2}{\rho^4}(\rho^2 + R^2) \quad \ldots\ldots\ldots\ldots \text{(i)}.$$

Again, $$\frac{F''}{v''^2} = \rho\left(\frac{d\theta'}{ds''}\right)^2 - \frac{d^2\rho}{ds''^2},$$

and $$\frac{F''(\rho^2 + R^2)}{v''^2} = \rho\left(\frac{d\theta'}{d\phi}\right)^2 - \frac{d^2\rho}{d\phi^2} = \rho - \frac{dR}{d\phi};$$

hence $$F'' = \frac{h''^2}{\rho^4}\left(\rho - \frac{dR}{d\phi}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

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Or, $$\frac{F''}{v''^2} = \frac{-ds \,.\, d\theta'}{ds''^2} - \frac{d^2\rho}{ds''^2} = \frac{1}{\rho}\left(\frac{ds}{ds''}\right)^2 - \frac{d^2\rho}{ds''^2}$$
$$= \frac{1}{\rho} - \frac{1}{\rho}\left(\frac{d\rho}{ds'}\right)^2 - \frac{d^2\rho}{ds''^2} \quad \ldots\ldots\ldots\ldots \text{(iii.)}$$

83. From equation (μ) we see that, if the times of describing the three arcs be the same, then $v''^2 = v^2 + v'^2$.

84. We may observe that by means of Art. 57 we may conversely, when we can solve the equations, find the primitive corresponding to a given radial.

85. From our definition it readily follows that the sum of the perimeters of all the circles of curvature of the primitive $= 2\pi\Sigma(r)$, and the sum of the areas of the same circles $= \frac{1}{2}\pi\Sigma(r^2)$, the summation being taken between the proper limits.

86. We venture to call the reader's attention to two points upon which we are disposed to think some light may be thrown by the preceding discussion, viz., properties of elliptic arcs and differential equations. We have not arrived at any satisfactory results in these two directions, but leave the examination to more competent hands.

Application of Radials to Steiner's Envelope.

1. The Envelope in Quest. 1679 has for some time engaged my attention, and I had obtained some interesting properties, besides Quest. 1649, before I was aware of its having been discussed elsewhere. My present object is more particularly to give such results as readily follow from the application of my method of *Radials* to the Envelope.

2. Reference is made throughout to the Abridgement of Steiner's Paper (*Reprint*, Vol. III., pp. 97—100) and to my articles in the *Reprint*, Vol. I., pp. 16—19, and Vol. II., pp. 27—38.

3. Take for axes a line through S parallel to AB and Sγ, and let $\angle$ AS$p = \theta$, then the equation to G will be

$$Y - R\cos C = \cot(B + \tfrac{1}{2}\theta)\left\{X - R\sin(C-\theta)\right\} \ldots\ldots\ldots \text{(i.)};$$

hence, by (α) and (β) of *Radials*, we have

$$\cot\theta' = \frac{dy}{dx} = \cot(B + \tfrac{1}{2}\theta),$$

$$(r'\sin^3\theta')^{-1} = \frac{d^2y}{dx^2} = \left\{4R\sin^3\theta'\sin(3\theta' - 2B - C)\right\}^{-1};$$

hence $r' = 4R\sin(3\theta' - 2B - C)$ is the radial of the envelope of G; or, turning the axes through a right angle, $r' = 4R\cos(3\theta' + A - B)$; but this, by Art. 17, and Todhunter's *Integral Calculus*, § 112, is the radial to a tricusp, a vertex of the curve lying on a line inclined to Sγ at an angle $= -\frac{1}{3}(A - B)$; and its intrinsic equation referred to a vertex is $s = \frac{4}{3}R\sin 3\theta$, whence we readily derive (9) the whole length of the curve $= 8R = 16r$.

4. If ρ, ρ' be the radii of curvature at t, t', and s_1, s_2 the corresponding arcs measured from a vertex, we have, from (5),

$$st = \tfrac{1}{2}\rho' = \tfrac{3}{2}s_1, \quad st' = \tfrac{1}{2}\rho = \tfrac{3}{2}s_2, \quad \rho + \rho' = 3(s_1 + s_2).$$

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Analogous properties may be easily established, by means of Art. 17, for any epicycloid or hypocycloid.

5. By means of (3), we see that (m_2) touches the curve, and that at its vertices; hence, by Art. 3 above, the cuspidal tangent through m is inclined to $S\gamma$ at the angle $\frac{1}{3}(A-B)$, which gives us the point w, as in (8), for a vertex.

6. The result of Quest. 1649, which I have proved geometrically on p. 59 of Vol. II. of the *Reprint*, may be proved by combining the equation to G′ with (i).

7. Since the coordinates of p are $R\sin(C-\theta)$, $R\cos(C-\theta)$, the equation to a perpendicular from P on G is

$$Y-R\cos(C-\theta) = -\tan(B+\tfrac{1}{2}\theta)\left\{X-R\sin(C-\theta)\right\} \quad \ldots\ldots \text{(ii.)},$$

the radial for the envelope of which is $r' = 8R\sin(3\theta'+A-B)$, or turned through a right angle $r' = 8R\cos(3\theta'+A-B)$, that is, a tricusp G'_3 similar to G_3 having its cuspidal tangents parallel to those of G_3; hence the envelope of *the axes of a system of parabolas which touch the sides of a triangle is a tricusp* having its parameter twice that of G_3.

8. Now, if we proceed to find the equation to G″, the foot-perpendicular line for the variable triangle ABp with regard to the point C, we see that i tis

$$Y-R\cos C = -\cot(B+\tfrac{1}{2}\theta)\left\{X-R\sin(A-B)\right\} \quad \ldots\ldots \text{(iii.)};$$

hence G and G″ are supplementally inclined to AB and intersect upon (m_2) in the point μ. By Quest. 1431 (p. 35, Vol. I. of the *Reprint*), the tangents at the vertices of the four parabolas, which have three points on a circle for intersection of tangents and the fourth for a focus, pass through a point (μ); and the locus of μ, by the above, is for three fixed points the nine-point circle of the triangle formed by them.

9. Also we see that the nine-point circles of the four triangles pass through the same point μ, and have, by Quest. 1431,

$$(\mu A)^2+(\mu B)^2+(\mu C)^2+(\mu p)^2 = D^2,$$

as we may very readily prove, by the above property, if we draw the figure.

10. The perpendiculars from p and p' on G, G′, respectively, manifestly intersect at right angles on (S); hence (S) is situated with respect to G'_3 in the same way that (m_2) is to G_3, and a little examination shows us that the vertices of G'_3 are situated on lines through S parallel to mw, &c., towards the cusps of G_3.

11. Similar properties, of course, obtain for G'_3 as for G_3, the cusps lying on a circle, centre S, radius 3R, &c.

12. If Aa, Bb, Cc, touch G_3 in L_1, L_2, L_3, then

$$DL_1\sin A = DL_2\sin B = DL_3\sin C = 2R\sin A\sin B\sin C.$$

N.B.—The subject of Radial Curves first occupied the writer's attention in February, 1863, and he was induced to work further at these curves by the persuasion of his able friend the Editor, who from the outset expressed his belief in the utility of the method.

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1711. (Proposed by Professor SYLVESTER.)—To a uniform beam supported at its extremities is suspended a weight equal to that of the beam; show that, if the point of suspension is nearer to the centre than to an end of the beam, the moment of strain will be greatest at that point, but otherwise greatest at a point at the same distance from the centre, as the point of suspension is from the end of the beam.

Solution by the REV. R. TOWNSEND, M.A.

The answer to this question follows immediately from the following principles :—

I. *A horizontal beam* AB, *supported at both extremities* A *and* B, *is loaded with a weight* E *at an arbitrary point* P; *the bending moment, due to the action of* E, *at any point* X *of its length, is* $\frac{E}{AB}$. BP . AX, *or* $\frac{E}{AB}$. AP . BX, *according as* X *is situated between* A *and* P, *or between* B *and* P.

For, the beam AB being held in constrained equilibrium by the weight E at the point P and by the two reactions $E \cdot \frac{BP}{AB}$ and $E \cdot \frac{AP}{AB}$ at its two extremities A and B, the entire bending moment at any point X of its length is consequently that arising from the reaction at the extremity A or B between which and the point P it lies.

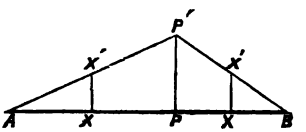

Fig 1.

COR. 1.—It follows from the above, that, for different positions of X along the beam, the bending moment, which is of course evanescent at each extremity A and B, increases uniformly from each to the point P, where it is the maximum, and has for value $\frac{E}{AB}$. AP . BP.

COR. 2.—The following construction for representing geometrically the magnitude of the bending moment, at any point X of the beam, follows also from the above. At the point of application P of the load E, draw a perpendicular PP′ to represent on any scale the magnitude of the bending moment at that point, and join AP′ and BP′; then the perpendicular XX′ at any point X, terminated by AP′ or BP′, will represent on the same scale its magnitude at that point.

II. *A horizontal beam* AB, *supported at both extremities* A *and* B, *is loaded with a number of weights* E, F, G, H, *&c., arbitrarily applied at different points* P, Q, R, S, *&c.; the entire bending moment, due to their combined action, at any point* X *of its length, is*

$$\frac{1}{AB}\left\{\Sigma(E \cdot AP) \cdot BX + \Sigma(F \cdot BQ) \cdot AX\right\};$$

the first summation comprising all the points of application between A *and* X, *and the second all between* B *and* X.

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This follows at once from Prop. I.; each term of each summation being, by it, the part of the entire moment due to the corresponding weight.

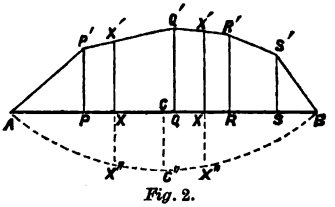

Fig. 2.

Cor. 1.—It follows from the above, that, during the passage of X along the beam from either extremity A to the other B, the bending moment, which of course vanishes at each extremity, varies uniformly during the passage from each weight to the next; that in the passage over each weight it undergoes an abrupt change; that it always attains its greatest value as some one of them is reached; and that, if it have equal values for any two contiguous weights, it has a constant value equal to each for all positions of X intermediate between them.

Cor. 2.—The following construction for representing geometrically the magnitude of the bending moment, at any point X of the beam, follows also from the above, and shows very clearly the several particulars just enumerated in Cor. 1. At the several points of application P, Q, R, S, &c., of the several loads E, F, G, H, draw the several perpendiculars PP′, QQ′, RR′, SS′, &c., to represent on any common scale the several magnitudes of the bending moment at those points, and join successively AP′, P′Q′, Q′R′, R′S′, &c., and finally S′B; then the perpendicular XX′, drawn from any point X (between P and Q suppose) and terminated by the corresponding connector (P′Q′ in that case), will represent on the same scale its magnitude at that point.

N.B.—If the entire load, in place of being applied in detached portions E, F, G, H, &c., at isolated points P, Q, R, S, &c., of the beam, were distributed continuously along its entire length according to any law, the polygon AP′Q′R′S′ . . . B in the above construction would then be a continuous curve, and the bending moment, which would then vary continuously along its entire length, would be represented at any point X by the ordinate XX′ to that curve. The following is the simplest case of such a distribution.

III. *A horizontal beam* AB, *supported at both extremities* A *and* B, *is loaded with a weight* W *distributed uniformly along its entire length; the entire bending moment due to its action at any point* X *is* $\frac{\frac{1}{2}W}{AB} . AX . BX$; *that is (by Prop.* I.) *equal to half of what it would be if the entire load were applied at that point.*

For, in the general formula for the bending moment at X (given for any law of distribution in Prop. II.), the two sums $\Sigma(E . AP)$ and $\Sigma(F . BQ)$, which are then of course integrals, have respectively for values $\frac{W}{AB} . \frac{AX^2}{2}$ and $\frac{W}{AB} . \frac{BX^2}{2}$; therefore the bending moment itself has for value

$$\frac{\frac{1}{2}W}{AB^2} . (AX^2 . BX + BX^2 . AX),$$

which, since $AX + BX = AB$, is manifestly equivalent to the above.

Cor. 1.—It follows from the above, that, as X traverses the length of the beam from one extremity to the other, the bending moment, which, as in

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every other case, vanishes of course at each extremity, varies continuously throughout the passage, has equal values at all pairs of points equidistant from its middle point C, and has there its greatest value of which the magnitude is $\frac{1}{8}W \cdot AB$.

Cor. 2.—The following construction for representing geometrically the bending moment at any point X follows also from the above. At the middle point C (Fig. 2) of the beam AB draw the perpendicular CC″, to represent on any scale the magnitude of the bending moment at that point, and with C″ as vertex describe on AB as chord the arc AC″B of the parabola thus determined (see the dotted line in Fig. 2); then the ordinate XX″ to that curve at any point X will represent on the same scale its magnitude at that point.

IV. *A uniform horizontal beam* AB, *supported at both extremities* A *and* B, *is loaded at any point* P *with a weight* W *equal to its own; the entire bending moment due to the combined weights of the load and beam is greatest at the point of application* P *if it be nearer to the centre* C *than to either extremity* A *or* B *of the beam, but otherwise greatest at the point* X, *whose distance* CX *from the centre* C *in the direction of* P *is equal to the distance* AP *of* P *from the nearer extremity* A *of the beam.* (Sylvester's Question.)

For since, for any point X between P and C, where the point of greatest moment must evidently lie, the portions of the bending moment due to the weights of the load and beam are, by Props. I. and III., respectively, $\frac{W}{AB} \cdot AP \cdot BX$ and $\frac{\frac{1}{2}W}{AB} \cdot AX \cdot BX$;

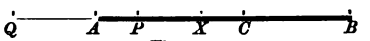

Fig. 3.

if Q be the point on AB produced at which $QA = 2AP$, the sum of the two, that is, the entire moment, is $\frac{\frac{1}{2}W}{AB} \cdot QX \cdot BX$; which is the greatest possible when X, consistently with the restriction of being between P and C, is as near as possible to the middle point of QB.

N.B.—The general problem of which the above is a particular case, viz., "For a uniform beam AB of given weight W, loaded with given weights E, F, G, H, &c., at given points P, Q, R, S., &c., to determine the point (or points) of maximum bending moment under the joint actions of its own and of the appended weights," may be solved readily by aid of the two constructions of Fig. 2 for the bending moments. For, the several perpendiculars PP′, QQ′, RR′, SS′, CC″ being drawn (as explained in Props. II. and III.) to represent on any common scale the several bending moments at P, Q, R, S, &c., due to the combined action of E, F, G, H, &c., at those points, and that at C due to the separate action of W; and the polygon AP′Q′R′S′.... B and parabola AC″B being constructed (as there also explained); every point X, the double ordinate X′X″ at which meets the parabola at a point X″ at which the tangent is parallel to the corresponding side (P′Q′ suppose) of the polygon, is evidently a point at which the bending moment (represented always by that ordinate) is a maximum; and so also, for the same reason, is every point of application (Q suppose) of an appended weight, the tangent at the intersection Q″ of whose ordinate with the curve determines with the corresponding angle P′Q′R′ of the polygon a triangle to which that angle is internal. These two cases evidently correspond respectively to the second and first cases in Professor Sylvester's Question.

Of course, between every two consecutive positions of maximum moment (when more than one exist) a position of minimum moment lies. Every such position corresponds evidently to a point of application (R suppose) of an

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appended weight for which the tangent at the corresponding point R″ of the parabola determines with the corresponding angle Q′R′S′ of the polygon a triangle to which the vertically opposite angle is internal; positions of minimum corresponding to those of maximum moment of the first species above noticed evidently not existing from the nature of the case.

1744. (Proposed by W. S. BURNSIDE, B.A.)—It is required to find (x_1, y_1, z_1), functions of (x, y, z), such that we may have identically

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1} = \frac{x^3+y^3+z^3}{xyz}.$$

I. *Solution by* PROFESSOR CAYLEY.

The Solution is in fact given in my "Memoir on Curves of the Third Order," *Philosophical Transactions*, vol. 147 (1857), pp. 415—446.

Write $\frac{x^3+y^3+z^3}{xyz} = -6l$; then, taking (X, Y, Z) as current coordinates, (x, y, z) are, it is clear, the coordinates of a point on the cubic curve $X^3+Y^3+Z^3+6lXYZ=0$; and if (x_1, y_1, z_1) are the coordinates of any other point on the same cubic curve, then we shall have

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1} = -6l = \frac{x^3+y^3+z^3}{xyz},$$

so that (x_1, y_1, z_1) will satisfy the condition in question. Hence, if from a given point (x, y, z) on the cubic curve we obtain by any geometrical construction another point on the curve, the coordinates of this new point will be functions (and, if the construction is such as to lead to a single point only, they will be *rational* functions) of (x, y, z), satisfying the condition in question.

For instance, if the point (x, y, z) be joined with any point (α, β, γ) on the curve, the joining line will again meet the curve in a single point, which may be taken to be the point (x_1, y_1, z_1). This assumes that we know on the cubic curve a point (α, β, γ); but such a point at once presents itself, viz., we may write $(\alpha, \beta, \gamma) = (1, -1, 0)$; which gives only the self-evident solution $(x_1, y_1, z_1) = (y, x, z)$. The point $(1, -1, 0)$ is clearly one of the nine points of inflexion of the cubic curve, and by using these in any manner whatever, viz., joining the point (x, y, z) with any point of inflexion, and then the new point with any other point of inflexion, and so on indefinitely, we obtain in connexion with the given point (x, y, z) seventeen other points on the curve, in all a system of eighteen points: these are

(x, y, z), $(x, \omega y, \omega^2 z)$, $(x, \omega^2 y, \omega z)$	(x, z, y), $(x, \omega z, \omega^2 y)$, $(x, \omega^2 z, \omega y)$
(y, z, x), $(\omega y, \omega^2 z, x)$, $(\omega^2 y, \omega z, x)$	(z, y, x), $(\omega z, \omega^2 y, x)$, $(\omega^2 z, \omega y, x)$
(z, x, y), $(\omega^2 z, x, \omega y)$, $(\omega z, x, \omega^2 y)$	(y, x, z), $(\omega^2 y, x, \omega z)$, $(\omega y, x, \omega^2 z)$

possessing remarkable geometrical properties; and of course each of the seventeen new points furnishes a (self-evident) solution of the given identity.

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But we may take $(\alpha, \beta, \gamma) = (x, y, z)$; the point (x_1, y_1, z_1) is here the point of intersection of the cubic by the tangent at the point (x, y, z); or say it is the "tangential" of the point (x, y, z). The values thus obtained for (x_1, y_1, z_1) are

$$(x_1, y_1, z_1) = \left\{x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)\right\},$$

which (excluding the above mentioned self-evident solutions) is in fact the most simple solution of the proposed identity. In order to verify that the last mentioned values of (x_1, y_1, z_1) are in fact the coordinates of the tangential of (x, y, z), I observe that this will be the case if only we have

$(x^2+2lyz)x_1+(y^2+2lzx)y_1+(z^2+2lxy)z_1 = 0$, $x_1^3+y_1^3+z_1^3+6lx_1y_1z_1 = 0$,

the first of which is obviously satisfied by the values in question; and for the verification of the second equation,

$$\begin{aligned} x_1^3+y_1^3+z_1^3 &= x^3(y^3-z^3)^3+y^3(z^3-x^3)^3+z^3(x^3-y^3)^3 \\ &= -x^9(y^3-z^3)-y^9(z^3-x^3)-z^9(x^3-y^3) \\ &= (x^3+y^3+z^3)(y^3-z^3)(z^3-x^3)(x^3-y^3), \\ x_1y_1x_1 &= xyz(y^3-z^3)(z^3-x^3)(x^3-y^3), \end{aligned}$$

$\therefore x_1^3+y_1^3+z_1^3+6lx_1y_1z_1 = (x^3+y^3+z^3+6lxyz)(y^3-z^3)(z^3-x^3)(x^3-y^3) = 0$

if $x^3+y^3+z^3+6lxyz = 0$; the same equations verify at once the identity

$$\frac{x_1^3+x_1^3+z_1^3}{x_1y_1z_1} = \frac{x^3+y^3+z^3}{xyz}.$$

Another solution is as follows: viz., if we take the third intersection with the cubic of the line joining the points (y, x, z) and $\left\{x(y^3-z^3), y(z^3-x^3), z(x^3-y^3)\right\}$, the coordinates of the line in question are

$$\begin{aligned} x_1 : y_1 : z_1 = \quad & x^6y^3+y^6z^3+z^6x^3-3x^3y^3z^3 \\ & : x^3y^6+y^3z^6+z^3x^6-3x^3y^3z^3 \\ & : xyz(x^6+y^6+z^6-y^3z^3-z^3x^3-x^3y^3). \end{aligned}$$

According to a very beautiful theorem of Professor Sylvester's in relation to the theory of cubic curves, the coordinates of a point which depends linearly on a given point of the curve are necessarily rational and integral functions *of a square degree* of the coordinates (x, y, z) of the given point; and moreover that (considering as *one* solution those which can be derived from each other by a mere permutation of the coordinates, or change of x into ωx &c.), there is only one solution of a given square degree m^2; the solutions of the degrees 4 and 9 are given above. The tangential of the tangential, or second tangential of the point (x, y, z), gives the solution of the degree 16; joining this second tangential with the original point (x, y, z), we have the solution of the degree 25; and the same solution is also given as the sixth point of intersection with the cubic, of the conic of 5-pointic intersection at the point (x, y, z). See my memoir "On the conic of 5-pointic contact at any point of a plane curve," *Philosophical Transactions*, vol. 149 (1859), pp. 371—400.

Addition to the foregoing Solution. On a system of Eighteen Points on a Cubic Curve.

Considering the cubic curve $x^3+y^3+z^3+6lxyz = 0$, we have the nine points of inflexion, which I represent as follows:—

$$\begin{array}{lll} a = (0, 1, -1), & d = (-1, 0, 1), & g = (1, -1, 0), \\ b = (0, 1, -\omega), & e = (-\omega, 0, 1), & h = (1, -\omega, 0), \\ c = (0, 1, -\omega^2), & f = (-\omega^2, 0, 1), & i = (1, -\omega^2, 0), \end{array}$$

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viz., ω being an imaginary cube root of unity, the coordinates of a are $(0, 1, -1)$, those of b, $(0, 1, -\omega)$, &c.

The points of inflexion lie (as is known) by threes on twelve lines; viz., the lines are

abc,	afh,	bfg,	cfi,
adg,	bdi,	cdh,	def,
aei,	beh,	ceg,	ghi.

Consider now a point on the curve, the coordinates whereof are (x, y, z), where of course $x^3+y^3+z^3+6lxyz=0$; this is one of a system of eighteen points on the curve, which may be represented as follows:—

$A=(x, y, z)$,	$D=(x, \omega y, \omega^2 z)$,	$G=(x, \omega^2 y, \omega z)$,
$B=(y, z, x)$,	$E=(\omega y, \omega^2 z, x)$,	$H=(\omega^2 y, \omega z, x)$,
$C=(z, x, y)$,	$F=(\omega^2 z, x, \omega y)$,	$I=(\omega z, x, \omega^2 y)$,
$J=(x, z, y)$,	$M=(x, \omega z, \omega^2 y)$,	$P=(x, \omega^2 z, \omega y)$,
$K=(z, y, x)$,	$N=(\omega z, \omega^2 y, x)$,	$Q=(\omega^2 z, \omega y, x)$,
$L=(y, x, z)$,	$O=(\omega^2 y, x, \omega z)$,	$R=(\omega y, x, \omega^2 z)$,

viz., the coordinates of A are (x, y, z); those of B are (y, z, x), &c.

The tangent at A meets the curve in a point, "the tangential of A," the coordinates whereof are $x(y^3-z^3)$, $y(z^3-x^3)$, $z(x^3-y^3)$; which point may be called A′. And we have thus the eighteen tangentials

A′, B′, C′, D′, E′, F′, G′, H′, I′, J′, K′, L′, M′, N′, O′, P′, Q′, R′.

The eighteen points A, B, &c., have the following property; viz., the line joining any two of them meets the cubic in a third point, which is either one of the nine points of inflexion, or one of the eighteen tangentials; there are through each point of inflexion 9 such lines, and through each tangential 4 such lines; $(9\times 9)+(18\times 4)=153=\frac{1}{2}(18.17)$, the number of pairs of points AB, AC, &c. The lines through the inflexions are the 81 lines obtained by joining any one of the points (A, B, C, D, E, F, G, H, I) with any one of the points (J, K, L, M, N, O, P, Q, R), as shown in the following Table:—

	A	B	C	D	E	F	G	H	I
J	a	d	g	c	f	i	b	e	h
K	d	g	a	f	i	c	e	h	b
L	g	a	d	i	c	f	h	b	e
M	c	f	i	b	e	h	a	d	g
N	f	i	c	e	h	b	d	g	a
O	i	c	f	h	b	e	g	a	d
P	b	e	h	a	d	g	c	f	i
Q	e	h	b	d	g	a	f	i	c
R	h	b	e	g	a	d	i	c	f

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viz., the line AJ passes through a, the line AK through d, &c.; the proof that AJ passes through a depends on the identical equation

$$\begin{vmatrix} x, & y, & z \\ x, & z, & y \\ 0, & 1, & -1 \end{vmatrix} = 0;$$

and the like for the other lines AK, AL, &c.

The lines through the tangentials are the 36 lines obtained by joining any two of the points (A, B, C, D, E, F, G, H, I) and the 36 lines obtained by joining any two of the points (J, K, L, M, N, O, P, Q, R); and these 72 lines pass through the tangentials, as shown by the table

ABC,	BDI,	CEG,	JKL,	KMR,	LNP,
ADG,	BEH,	CFI,	JMP,	KNQ,	LOR,
AEI,	BFG,	DEF,	JNR,	JOP,	MNO,
AFH,	CDH,	GHI,	JOQ,	LMQ,	PQR,

viz., in the triad ABC, BC passes through A′, CA through B′, AB through C′; and the like for the other triads. The proof that BC passes through A depends on the identical equation

$$\begin{vmatrix} y, & z, & x \\ z, & x, & y \\ x(y^3-z^3), & y(z^3-x^3), & z(x^3-y^3) \end{vmatrix} = 0;$$

and the like for the other combinations of points.

If we attend only to the points A, B, C and their tangentials A′, B′, C′; then we have on the cubic three points A, B, C, such that the line joining any two of them passes through the tangential of the third point. And the figure may be constructed by means of the three real points of inflexion a, d, g, as follows, viz., joining these with any point J on the cubic, the lines so obtained respectively meet the cubic in three new points which may be taken for the points A, B, C. Or if one of these points, say A, be given, then joining it with one of the three real inflexions, this line again meets the cubic in the point J, and from it by means of the other two real inflexions we obtain the remaining points B and C; it is clear that, A being given, the construction gives three points, say J, K, L, each of them leading to the same two points B and C.

We may consider the question from a different point of view. Let A, B, C be given points, and let there be given also three lines passing through these three points respectively; through the given points, touching at these points the given lines respectively, describe a cubic; and let the given lines again meet the cubic in the points A′, B′, C′ respectively. The equation of the cubic contains three arbitrary parameters; but when two of these are properly determined, the points A, B, C and their tangentials A′, B′, C′ will be related as in the theorem; viz., the line through any two of the points will pass through the tangential of the third point. The analytical investigation is as follows:—

Let the equations of the three tangents be $x = 0$, $y = 0$, $z = 0$, and suppose that, for the points A, B, C respectively, we have

$$(x = 0,\ y = \lambda z),\ (y = 0,\ z = \mu x),\ (z = 0,\ x = \nu y),$$

then the equation of a cubic touching the three lines at the three points respectively will be

$$(y-\lambda z)^2\,(\nu^2 \mathrm{B}y + \mathrm{C}z) + (z-\mu x)^2\,(\lambda^2 \mathrm{C}z + \mathrm{A}x) + (x-\nu y)^2(\mu^2 \mathrm{A}x + \mathrm{B}y)$$
$$-\mu^2 \mathrm{A}x^3 - \nu^2 \mathrm{B}y^3 - \lambda^2 \mathrm{C}z^3 + \mathrm{K}xyz = 0,$$

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where A, B, C, K are arbitrary coefficients; but if $A : B : C = \lambda : \mu : \nu$, then the equation is

$$(y-\lambda z)^2\,\nu\,(\mu\nu y+z)+(z-\mu x)^2\,\lambda\,(\nu\lambda z+x)+(x-\nu y)^2\,\mu\,(\lambda\mu x+y)$$
$$-\lambda\mu^2x^3-\mu\nu^2y^3-\nu\lambda^2z^3+Kxyz=0,$$

where

A, A′ are the intersections of $x=0$, by $y-\lambda z=0$, $\mu\nu y+z=0$ respectively,
B, B′ „ „ $y=0$, „ $z-\mu x=0$, $\nu\lambda z+x=0$ „
C, C′ „ „ $z=0$, „ $x-\nu y=0$, $\lambda\mu w+y=0$ „

the equations of BC, CA, AB thus are

$$-\mu x+\mu\nu y+z=0,\quad x-\nu y+\nu\lambda z=0,\quad \lambda\mu x+y-\lambda z=0,$$

which pass through A′, B′, and C′, respectively.

If we consider along with the points A, B, C the points J, K, L, and their respective tangentials, then we have inscribed in the cubic a hexagon ALBJCK which has the following properties, viz., the pairs of opposite sides and the three diagonals pass through the three real inflexions *in line*, viz.,

AL,	JC,	BK, through	g
LB,	CK,	JA, „	a
BJ,	KA,	CL, „	d.

This shows that the six points A, B, C, J, K, L are the intersections of the cubic by a conic; and moreover, considering the triangles ABC, JKL formed by the alternate vertices, then in each triangle the sides pass through the tangentials of the opposite vertices respectively.

In what precedes we have in effect found the coordinates (z, x, y) of the third point of intersection with the cubic, of the line joining the points (y, z, x) and $\{x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)\}$. The coordinates of the same point may be otherwise found by a direct investigation, as follows:

Write

$$x_2 : y_2 : z_2 = x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3);\ x_1 : y_1 : z_1 = y : z : x.$$

If in the equation of the curve we substitute for x, y, z, the values ux_1+vx_2, uy_1+vy_2, uz_1+vz_2, we find

$$u\{x_1^2x_2+y_1^2y_2+z_1^2z_2+2l\,(x_2y_1z_1+y_2z_1x_1+z_2x_1y_1)\}$$
$$+v\{x_1x_2^2+y_1y_2^2+z_1z_2^2+2l\,(x_1y_2z_2+y_1z_2r_2+z_1x_2y_2)\}=0,$$

say $uP+vQ=0$; we may therefore write $u=Q$, $v=-P$, and the coordinates of the third point are Qx_1-Px_2, Qy_1-Py_2, Qz_1-Pz_2. Now

$$Qx_1-Px_2=y_1y_2(x_1y_2-x_2y_1)+z_1z_2(x_1z_2-x_2z_1)+2l(x_1^2y_2z_2-x_2^2y_1z_1)$$
$$=yz(z^3-x^3)\{y^2(z^3-x^3)-zx(y^3-z^3)\}$$
$$+zx(x^3-y^3)\{yz(x^3-y^3)-x^2(y^3-z^3)\}$$
$$+2l\{y^2\,.\,yz\,.\,(z^3-x^3)(x^3-y^3)-x^2(y^3-z^3)^2zx\}.$$
$$=(x^3y^6+y^3z^6+z^3x^6-3x^3y^3z^3)z$$
$$+xyz(x^6+y^6+z^6-y^3z^3-z^3x^3-x^3y^3)z$$
$$-2l(x^6y^3+y^6z^3+z^6x^3-3x^3y^3z^3)z;$$

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so that we have $Qx_1 - Px_2 = \Pi z$; and in like manner $Qy_1 - Py_2 = \Pi x$, $Qz_1 - Pz_2 = \Pi y$; and therefore $Qx_1 - Px_2 : Qy_1 - Py_2 : Qz_1 - Pz_2 = z : x : y$, which proves the theorem.

I consider in like manner the following question; viz., if (y, x, z) be joined with the tangential of (x, y, z); to find the third point of intersection. We have here

$$x_2 : y_2 : z_2 = x(y^3 - z^3) : y(z^3 - x^3) : z(x^3 - y^3); \quad x_1 : y_1 : z_1 = y : x : z;$$

and P, Q as before; and the coordinates of the third point are

$$Qx_1 - Px_2 : Qy_1 - Py_2 : Qz_1 - Pz_2;$$

also
$$\begin{aligned} Qx_1 - Px_2 = {} & xy(z^3 - x^3)\left\{y^2(z^3 - x^3) - x^2(y^3 - z^3)\right\} \\ & + z^2(x^3 - y^3)\left\{yz(x^3 - y^3) - zx(y^3 - z^3)\right\} \\ & + 2l\left\{y^2z(z^3 - x^3)(x^3 - y^3) - x^2(y^3 - z^3)^2 zx\right\} \\ = {} & x\left\{y^2(z^3 - x^3)^2 - z^2(x^3 - y^3)(y^3 - z^3)\right\} \\ & + y\left\{z^2(x^3 - y^3)^2 - x^2(y^3 - z^3)(z^3 - x^3)\right\} \\ & + 2lx\left\{y^2(z^3 - x^3)(x^3 - y^3) - x^2(y^3 - z^3)^2\right\} \end{aligned}$$

that is $Qx_1 - Px_2 = (x + y - 2lz)(x^2z^6 + y^2x^6 + z^2y^6 - 3x^2y^2z^2)$;

similarly $Qy_1 - Py_2 = (x + y - 2lz)(y^2z^6 + y^6x^2 + z^2x^6 - 3x^2y^2z^2)$;

also $Qz_1 - Pz_2 = (x + y - 2lz)(x^6 + y^6 + z^6 - y^2z^2 - x^2z^2 - x^2y^2)\, xyz$;

and we hence have the values

$$X : Y : Z = x^6y^3 + y^6z^3 + z^6x^3 - 3x^2y^2z^2 : x^3y^6 + y^3z^6 + z^3x^6 - 3x^3y^3z^3 : xyz(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3)$$

for the coordinates of the point in question.

II. *Solution by the* Proposer; Professor Cremona; S. Bills; *and others.*

Since $x(y-z) + y(z-x) + z(x-y) = 0$, we have

$$\left\{x(y-z) + y(z-x) + z(x-y)\right\}\left\{(y-z)^2 + (z-x)^2 + (x-y)^2\right\} =$$
$$\Sigma x(y-z)^3 - (y-z)(z-x)(x-y)\ \Sigma(y+z-x) = 0;$$

$$\therefore x(y-z)^3 + y(z-x)^3 + z(x-y)^3 = (x+y+z)(y-z)(z-x)(x-y).$$

Changing x, y, z into x^3, y^3, z^3, and putting x_1, y_1, z_1 for $x(y^3 - z^3)$, $y(z^3 - x^3)$, $z(x^3 - y^3)$, we have

$$x_1^3 + y_1^3 + z_1^3 = (x^3 + y^3 + z^3)(y^3 - z^3)(z^3 - x^3)(x^3 - y^3),$$

also
$$x_1y_1z_1 = xyz(y^3 - z^3)(z^3 - x^3)(x^3 - y^3),$$

therefore
$$\frac{x_1^3 + y_1^3 + z_1^3}{x_1y_1z_1} = \frac{x^3 + y^3 + z^3}{xyz}.$$

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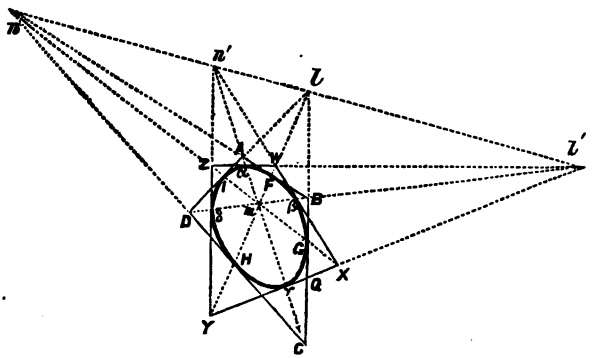
1751. (Proposed by Professor CAYLEY.)—Let ABCD be any quadrilateral. Construct, as shown in the figure, the points F, G, H, I: in BC find a point Q such that $\frac{BG}{BC} \cdot \frac{CQ}{GQ} = \frac{1}{\sqrt{2}}$; and complete the construction as shown in the figure. Show that an ellipse may be drawn passing through the eight points F, G, H, I, α, β, γ, δ, and having at these points respectively the tangents shown in the figure.



Solution by PROFESSOR CREMONA.

Conservons la figure de M. Cayley, et désignons, de plus, par des lettres les points (BC, AD) = l, (CA, BD) = m, (AB, CD) = n, (BD, ln) = l', (AC, ln) = n'. On sait, par les propriétés connues du quadrilatère complet (AC, BD, ln), que les systèmes (AB, Fn), (BC, Gl), (CD, Hn), (DA, Il) sont harmoniques; on sait en outre que quatre points pris dans les côtés d'un quadrilatère complet et tels qu'ils forment avec les ternes de sommets le même rapport anharmonique sur chaque côté, sont les points de contact d'une conique inscrite. Donc les droites AB, BC, CD, DA touchent en F, G, H, I une même conique; et pour cette conique le quadrilatère circonscrit est *harmonique*, parceque chaque côté est divisé harmoniquement par les trois autres et par le point de contact. Les points m, n' sont, par rapport à cette conique, les poles des droites ln, BD; donc la polaire de l' est mn' savoir AC; c'est-à-dire que les points α, γ, où la conique est touchée par les tangentes issues du point l', sont collinéaires avec mn'AC. De même les points β, δ où la conique est touchée par les tangentes issues de n' sont sur la droite ml'BD. Ces quatre tangentes issues de l' et n' forment un second quadrilatère circonscrit harmonique; car ex. g. les 4 points (WZ, $l'\alpha$) sont perspectifs aux 4 points (ln, $l'n'$) qui forment un système harmonique.

On peut observer encore que, des propriétés connues du quadrilatère complet (XZ, WY, $l'n'$), pour lequel le triangle diagonal est lmn, il suit évidemment que les droites $a\delta$, $\beta\gamma$ passent par l, et que les droites $a\beta$, $\gamma\delta$ passent par n; de même que l' est l'intersection de FI, GH, et n' est l'intersection de FG, HI.

Pour construire le point Q, duquel dépend le nouveau quadrilatère, calculons le rapport anharmonique (BQGC) = x. Le point Q étant un point double de l'involution (Bl, GC, ...), on aura l'égalité (BQGC) = (Ql GC), et par

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conséquent $(QI'GC) = x$. De cette égalité et de cette autre $(BI'GC) = \frac{1}{2}$, qui exprime l'harmonie du système (BC, GI'), on tire par la division $(BQGC) = \frac{1}{2x}$. Mais l'on a $(BQGC) = x$; donc $x^2 = \frac{1}{2}$, ce qui donne les deux points doubles de l'involution, c'est-à-dire les points où BC est coupée par les tangentes issues de I'.

[Professor CAYLEY remarks that if ABCD is the perspective representation of a square, then the ellipse is the perspective representation of the inscribed circle; the theorem gives eight points and the tangent at each of them; and the ellipse may therefore be drawn by hand with an accuracy quite sufficient for practical purposes. The demonstration is immediate, by treating the figure as a perspective representation: the gist of the theorem is the very convenient construction in perspective which it furnishes.]

1695. (Proposed by H. R. GREER, B.A.)—If u_m be a homogeneous function of the degree m in any number of independent variables, prove that

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dx} \ldots\right) \log u_m = m, \text{ identically;}$$

$$\left(x^2\frac{d^2}{dx^2} + y^2\frac{d^2}{dy^2} \text{ &c.} + 2xy\frac{d}{dx}\cdot\frac{d}{dy} \text{ &c.}\right) \log u_m = -m;$$

and continue the theorem.

Solution by F. D. THOMSON, M.A.

Let $f(x, y, z, \ldots) \equiv \log u_m$; then we have

$$f(x+hx, y+hy, \ldots) = \log\left\{u_m(1+h)^m\right\} = \log u_m + m\log(1+h)$$

but, by Taylor's Theorem,

$$f(x+hx, y+hy, \ldots) = \log u_m + h\left(x\frac{d}{dx} + y\frac{d}{dy} + \ldots\right)\log u_m + \frac{h^2}{1.2}\left(x\frac{d}{dx} + y\frac{d}{dy} + \ldots\right)^2 \log u_m + \text{&c.};$$

therefore $\log u_m + m\log(1+h) = \log u_m + h\left(x\frac{d}{dx} + \text{&c.}\right)\log u_m + \text{&c.};$

therefore $$m\left\{h - \frac{h^2}{2} + \frac{h^3}{3} - \ldots + (-)^{n+1}\frac{h^n}{n} + \ldots\right\} =$$

$$h\left(x\frac{d}{dx} + \ldots\right)\log u_m + \ldots + \frac{h^n}{\underline{|n}}\left(x\frac{d}{dx} + \ldots\right)^n \log u_m + .$$

hence, equating coefficients of h^n, we obtain

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + \ldots\right)\log u_m = (-)^{n+1} m\,\underline{|n-1}.$$

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1493. (Proposed by Dr. SALMON, F.R.S.)—If a conic have double contact with two fixed conics, find the envelope of its chords of intersection with another fixed conic having double contact with one of the two fixed conics.

Solution by the PROPOSER; *and* F. D. THOMSON, M.A.

Let the two fixed conics be $S = 0$, $S + xy = 0$; then the variable conic

$$k^2x^2 + 2k(2S + xy) + y^2 = 0, \quad \text{or} \quad (kx + y)^2 + 4kS = 0,$$

and its intersections with $S + x^2 = 0$ are $kx + y \pm 2x\sqrt{(k)} = 0$, whose envelope is $xy = z^2$.

If, however, instead of touching the conic $S + xy$, the condition be that the conic should pass through two fixed points, let the variable conic be $S - (\lambda x + \mu y + \nu z)^2$; then we have the two conditions

$$\lambda x' + \mu y' + \nu z' = \sqrt{(S')}, \quad \lambda x'' + \mu y'' + \nu z'' = \sqrt{(S'')};$$

whence it follows that the line

$$\lambda x + \mu y + \nu z \pm (ax + by + cz)$$

passes through one of four fixed points.

The reciprocals of these questions are as follows: viz., If a conic touch doubly two fixed conics, the locus of the intersection of common tangents with another also having double contact with one of the fixed conics is a conic; except the second fixed conic breaks up into right lines, when the locus also breaks up into right lines.

1721. (Proposed by G. O. HANLON.)—Find the locus of a point such that, if perpendiculars are drawn therefrom on the sides of a quadrilateral, a circle will pass through the feet of these perpendiculars; and show geometrically how to find points on the locus; also determine the quadrilateral in order that the locus may be a conic.

I. *Solution by* T. A. HIRST, F.R.S.

The problem is obviously equivalent to finding the locus of the foci of a system of conics inscribed in the given quadrilateral. Since there is but one conic of such a system which touches an arbitrary line, the locus required (see my Solution of Question 1717, *Reprint*, Vol. IV. p. 19) must be a *circular* cubic passing through the six intersections of the sides of the complete quadrilateral. In order that this locus may break up into a *conic* and a right line, the latter constituent must be at infinity, and on it must be situated, necessarily, a pair of opposite intersections of the quadrilateral. In other words, the latter must be a parallelogram.

II. *Solution by* W. S. BURNSIDE, B.A.; *the* PROPOSER; J. DALE; *and others.*

As it is well known that the feet of the perpendiculars from the foci of a

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conic on the tangents lie on a circle, the locus in question is that of the foci of conics touching four lines. Let the equations of the lines be $x = 0$, $y = 0$, $z = 0$, $ax + \beta y + \gamma z = 0$; then the equation of the conic is of the form

$$(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0, \text{ where } l\alpha^{-1} + m\beta^{-1} + n\gamma^{-1} = 0;$$

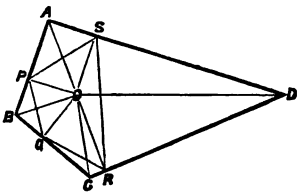
also (see the second Solution of Quest. 1717, *Reprint*, Vol. IV. p. 20.)

$$l : m : n = (y^2 + z^2 + 2yz\cos A)x : (z^2 + x^2 + 2zx\cos B)y : (x^2 + y^2 + 2xy\cos C)z;$$

hence the equation of the required locus is

$$(y^2 + z^2)\frac{x}{\alpha} + (z^2 + x^2)\frac{y}{\beta} + (x^2 + y^2)\frac{z}{\gamma} + 2xyz\left(\frac{\cos A}{\alpha} + \frac{\cos B}{\beta} + \frac{\cos C}{\gamma}\right) = 0.$$

[Points in the locus may be easily constructed from the following property. Let ABCD be the quadrilateral, O a point on the required locus, and P, Q, R, S the feet of the perpendiculars from O on the sides of the quadrilateral. Then $\angle AOB + COD = ASP + BQP + CQR + DSR = 2$ right-angles, since $PQR + PSR = 2$ right-angles. Hence if two circular segments be drawn on AB, CD, so as to contain supplemental angles, the arcs of these segments will intersect in points on the required locus. From the same property, moreover, the Cartesian equation of the locus may be readily obtained. In another form, the equation of the locus is given in Salmon's *Conics*, 4th ed., p. 261, ex. 15.]



1629. (Proposed by Dr. BOOTH, F.R.S.)—To investigate the geometrical signification of e, the base of the Napierian logarithms.

Solution by the PROPOSER.

Let $\Pi(m, \theta)$ denote the arc of a parabola, measured from the vertex to the point of the curve at which, a tangent being drawn, the focal perpendicular on it makes the angle θ with the axis. Let m be the focal distance of the vertex, or *modulus*, (we shall see presently the propriety of this definition,) then by an elementary formula for rectification, we shall have

$$\Pi(m, \theta) = m\sec\theta\tan\theta + m\int\sec\theta d\theta \ldots\ldots\ldots\ldots (1).$$

Since $m\sec\theta\tan\theta$ is the projection of the radius vector on the tangent, it may be called for brevity the *protangent*, and we have by common division,

$$\int\sec\theta d\theta = \int(1-\sin^2\theta)^{-1} d(\sin\theta) = \sin\theta + \tfrac{1}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + \tfrac{1}{7}\sin^7\theta + \&c. (2).$$

In (1), if we make m equal to unity, and assume that the difference between the parabolic arc and its protangent is also equal to unity, we shall have $\Pi(1, \theta) - \sec\theta\tan\theta = 1$, and therefore

$$\int\sec\theta\, d\theta = 1 = \sin\theta + \tfrac{1}{3}\sin^3\theta + \tfrac{1}{5}\sin^5\theta + \&c. \ldots\ldots\ldots (3)$$

and reversing this series by known methods, we shall have

$\sin\theta = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}}$, e being the number called the base of the Napierian

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logarithms; or, solving for e,

$$\sec\theta + \tan\theta = e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).*$$

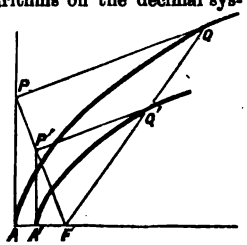
More generally, if we take the difference between the parabolic arc and its protangent equal to n, we shall have

$$\sec\theta_1 + \tan\theta_1 = e^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Hence, if on the modulus m of a parabola we describe a *Logocyclic* curve, or the curve whose polar equation is $r = m(\sec\theta \pm \tan\theta)$, and draw a tangent to the parabola perpendicular to this radius vector, the difference between the parabolic arc and its *protangent* will be the logarithm of the number $\sec\theta + \tan\theta$ and of its reciprocal $\sec\theta - \tan\theta$, when $m = 1$.

Hence also, if we describe a series of confocal and coaxal parabolas, these parabolas may represent by their arcs as many systems of logarithms. Thus, if we take the parabola whose modulus $m = \cdot 4342$, we shall have the parabola which represents the decimal system of logarithms.

Let D and Δ be the differences between the arcs of any two parabolas and their corresponding protangents to the same angle θ, the moduli being m and 1; then, as the confocal parabolas are similar figures and similarly placed, we shall have by similar triangles $D : \Delta = m : 1$. Draw the radius vector $\sec\theta + \tan\theta = 10$, and take the logarithms in the Napierian system; then $\Delta = \log 10$, and $D = 1$, since D is the decimal logarithm of the base 10; hence $m \log 10 = 1$. Again, if we take the logarithms on the decimal system, draw $\sec\theta + \tan\theta = e$; then $\Delta = 1$ and $D = \log e$, whence $m = \log e$. In the figure, since, FPQ and FP′Q′ are similar triangles, we have $AQ - QP : A'Q' - Q'P' = 1 : m$. Hence, also, as all these confocal parabolas are similar figures and are to each other as their parameters or moduli, it follows that all the logarithms, or tangential differences of the same number or logocyclic radius vector, are to each other as their moduli. And further, as the logocyclic curve admits of no negative radius vector, negative numbers can have no logarithms. And if we change $\sec\theta$ into $\cos\theta$, and $\tan\theta$ into $i\sin\theta$, we shall find the logocyclic changed into a circle whose radius is $r = \cos\theta + i\sin\theta$, whence it follows that *imaginary* numbers have circular logarithms, while real numbers have parabolic or real logarithms.



1677. (Proposed by C. TAYLOR, M.A.)—Prove that a chord of constant length, drawn through either point of contact of a circle which has double contact with a fixed conic, envelops a confocal conic.

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* Since $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.\right)$,

we have $\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = 2\left(\sin\theta + \frac{1}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + \&c.\right) = 2$,

or $\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = \frac{(1+\sin\theta)^2}{\cos^2\theta} = e^2$, or $\sec\theta + \tan\theta = e$.

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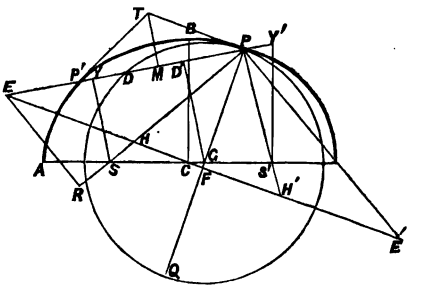
Solution by J. DALE.



Let CA, CB be the semi-axes of the given conic, S, S′ its foci; PDQ a circle having double contact with this conic, P being one of the points of contact; and PD a chord of given length. From G, the foot of the normal PG, which is the centre of the circle, draw GD′ perpendicular to PD, then PD′ ($= \frac{1}{2}$PD) is given. Through the centre C draw the diameter conjugate to CP (that is to say, parallel to the tangent at P) meeting PD, PS, PS′ in E, H, H′. Then, by known properties, PH = PH′ = CA; and from the similar triangles PGD′, PEF, we have PG : PD′ = PE : PF, therefore PE . PD′ = PG . PF = CB²; hence PE is of constant magnitude.

Draw SY, S′Y′ perpendicular to PE, and ER perpendicular to ES; then, by similar triangles, we have

PE : ER = PS : SY, ER : EH = PF : PH, ∴ PE : EH = PS . PF : SY . PH;

Similarly $PE : EH' = PS' . PF : S'Y' . PH'$;

therefore $PE^2 : EH . EH' = PS . PS' . PF^2 : SY . S'Y' . PH^2$.

But $EH . EH' = EF^2 - FH^2 = PE^2 - PH^2$;

also $PS . PS' . PF^2 = CA^2 . CB^2$, and $P H^2 = CA^2$;

therefore $PE^2 : PE^2 - CA^2 = CB^2 : SY . S'Y'$; hence the rectangle SY . S′Y′ is *constant*, and therefore PE is a tangent to a conic whose foci are S, S′. If CA′, CB′ are the semi-axes of this conic, we have

$$CB'^2 = S'Y . S'Y' = \frac{CB^2}{PE^2}(PE^2 - CA^2) = CB^2 - \left(\frac{PD . CA}{2CB}\right)^2,$$

$$CA'^2 = CA^2 - CB^2 + CB'^2 = \frac{CA^2}{PE^2}(PE^2 - CB^2) = CA^2 - \left(\frac{PD . CA}{2CB}\right)^2.$$

Let PE cut the fixed conic in P′, and at P′ draw a tangent meeting the tangent at P in T; then T lies on the normal to the enveloped conic, through the point of contact, and if TM be drawn perpendicular to PD, M is the point where PD touches the enveloped conic.

[The converse of this theorem, viz., "If PE be a tangent to a confocal conic, the line PE cut off by the parallel CE is of constant magnitude," forms Question 2026 of the *Lady's and Gentlemen's Diary* for 1864.]

1696. (Proposed by R. TUCKER, M.A.)—Supplemental chords of an ellipse are produced to meet the tangents at the extremities of the corresponding diameter; prove that the rectangle contained by the intercepts on the tangents is constant.

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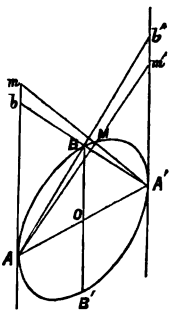
Solutions (1) *by* ARCHER STANLEY; (2) *by* ALPHA *and* J. DALE.

1. The connectors of A and A′ with the several points M of the conic constitute homographic pencils, and accordingly the points m, m', where these connectors intersect the parallel tangents at A and A′, constitute homographic ranges. To the point of contact of either of these tangents corresponds, manifestly, the point at infinity on the other. Hence, if BB′ be the diameter conjugate to AA′, and b, b' the points which correspond to B, the anharmonic ratios $[A\,\infty mb]$, $[\infty\, A'm'b']$ will be equal; whence we may at once deduce the relation



$$Am \cdot A'm' = Ab \cdot A'b' = (BB')^2 = \text{a constant.}$$

The above demonstration is essentially the same as that given by Chasles, at p. 119 of his *Traité des Sections Coniques*.

2. *Otherwise:* Referred to the conjugate semi-diameters OA′ ($= a$) and OB ($= b$) as axes, let the coordinates of M be (h, k); then the equations of AMm', A′Mm are

$$k(x+a) = (h+a)y \ldots . (Am'); \quad k(x-a) = (h-a)y \ldots . (A'm).$$

Making $x = a$ in (Am') and $= -a$ in $(A'm)$, we have

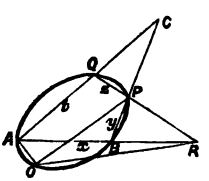
$$A'm' = \frac{2ak}{a+h}, \quad Am = \frac{2ak}{a-h}; \quad \therefore Am \cdot A'm' = \frac{4a^2k^2}{a^2-h^2}.$$

But $b^2h^2 + a^2k^2 = a^2b^2$, or $a^2k^2 = b^2(a^2-h^2)$; $\therefore Am \cdot A'm' = b^2$.

1586. (Proposed by X. U. J.)—If a straight line meet the sides BC, CA, AB of a triangle in P, Q, R respectively, the tangents at O to the conics OAPBQ, OAPCR form with the straight lines OA, OP a harmonic pencil. Also, if a fourth harmonic CK be drawn to the pencil CA, CO, CP, and a fourth harmonic RK to the pencil RA, RO, RP, these will intersect upon the tangent at O to the conic OAPBQ; and if a fourth harmonic be now drawn to KC, KO, KR, it will be *invariable* with respect to the three conics OAPBQ, OBQRC, OCRAP.

Solution by F. D. THOMSON, M.A.

Let $x = 0$, $y = 0$, $z = 0$, $t = 0$, be the equations to AB, BP, PQ, QA respectively, where (l, m, n, r being constants) x, y, z, t are connected by the equations



$$x+y+z+t = 0, \quad lx+my+nz+rt = 1.$$

Then the equation to a conic round ABPQ is of the form $xz = kyt$, and if this pass through the point O, whose coordinates are (x', y', z', t') suppose, we

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must have $x'z' = ky't'$; hence the equation to the conic OABPQ is $\frac{xz}{x'z'} = \frac{yt}{y't'}$; and therefore the equation to the tangent at O is

$$\frac{x}{x'} + \frac{z}{z'} = \frac{y}{y'} + \frac{t}{t'}, \text{ or } \left(\frac{x}{x'} - \frac{t}{t'}\right) - \left(\frac{y}{y'} - \frac{z}{z'}\right) = 0 \ldots\ldots\ldots (\alpha).$$

Again, the equation to the conic OAPCR is $\frac{xy}{x'y'} = \frac{zt}{z't'}$; and therefore the equation to the tangent at O is

$$\frac{x}{x'} + \frac{y}{y'} = \frac{z}{z'} + \frac{t}{t'}, \text{ or } \left(\frac{x}{x'} - \frac{t}{t'}\right) + \left(\frac{y}{y'} - \frac{z}{z'}\right) = 0 \ldots\ldots\ldots (\beta).$$

Now $\frac{x}{x'} - \frac{t}{t'} = 0$ is the equation to OA, and $\frac{y}{y'} - \frac{z}{z'} = 0$ is the equation to OP; hence, by (α) and (β) we see that OA, OP, and the tangents at O form a harmonic pencil.

Again, the equation to CO is $\frac{y}{y'} - \frac{t}{t'} = 0$,

herefore the equation to CK is $\frac{y}{y'} + \frac{t}{t'} = 0$;

similarly the equation to RK is $\frac{x}{x'} + \frac{z}{z'} = 0$;

therefore, by (α), K is a point on the tangent at O to the conic OABPQ. The fourth harmonic to KC, KO, KR will be the line

$$\left(\frac{x}{x'} + \frac{z}{z'}\right) + \left(\frac{y}{y'} + \frac{t}{t'}\right) = 0, \text{ or } \frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} + \frac{t}{t'} = 0 \ldots\ldots (\gamma).$$

This equation (γ) involves x, y, z, t symmetrically, and would therefore have been arrived at if we had started with either of the conics OBQCR or OCRAP.

1707. (Proposed by T. Cotterill, M.A.)—If A, B, C are the distances of a line from the angles of a triangle which it cuts in the points d, e, f respectively; then

$$\frac{A(B-C)}{ef} = \frac{B(C-A)}{fd} = \frac{C(A-B)}{de} = \frac{(B-C)(C-A)(A-B)}{2k},$$

k being the area of the triangle. The envelope of the lines in which $\lambda . ef + \mu . fd + \nu . fe = 0$ is a parabola; λ, μ, ν being as the coordinates of its axis. This equation can be written in such shapes, that an orthogonal substitution gives the envelope of the tangent at the vertex, and their form shows that they are also the envelope of the asymptotes of the rectangular hyperbolas circumscribing the triangle. The intercept of such a tangent between two sides of the triangle is equal to the projection of the remaining side upon it.

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Solution by the PROPOSER.

Let a, b, c be the inclinations of the line (A, B, C) to the sides a, b, c. In any triangle, the product of a side and the sines of the adjacent angles is equal to the product of the perpendicular upon it from the opposite angle and the sine of that angle. Applying this to the triangles Aef, Bfd, Cde, we have $ef \sin b \sin c = \sin A \,.\, A$; $fd \sin c \sin a = \sin B \,.\, B$; $de \sin a \sin b = \sin C \,.\, C$. But $B-C = a \sin a$; $C-A = b \sin b$; $A-B = c \sin c$.

Hence eliminating the sines, and reducing, we have

$$\frac{ef}{A(B-C)} = \frac{fd}{B(C-A)} = \frac{de}{C(A-B)} = \frac{2k}{(B-C)(C-A)(A-B)}.$$

Among the many applications which may be made of this formula, let us consider the equation $\lambda \,.\, ef + \mu \,.\, fd + \nu \,.\, de = 0$, or for shortness $\Sigma\lambda \,.\, ef = 0$, where λ, μ, ν are constant, and therefore as the ordinates of a certain line ($A_m B_m C_m$). Then the equation becomes $\Sigma A_m A(B-C) = 0$, or $\Sigma a \sin am \,.\, BC = 0$, from which, if $B+C-A = X$, &c., we obtain $\Sigma a \sin am \,.\, X^2 = 0$, the equations to a parabola referred, 1st, to a circumscribing; 2ndly, to a self-conjugate triangle. By taking the emanant, it will be seen that the point of contact with the line infinity is in the direction of m; i.e. m may be taken as the axis. Other forms arise from observing that $\Sigma A_m \,.\, ef = 0$ and $\Sigma ef = 0$ give the equations $\dfrac{ef}{a \sin am} = \dfrac{fd}{b \sin bm} = \dfrac{de}{c \sin cm}$. Combining these equations with the values of ef, fd, de, just given, we have

$$\frac{a \sin a \,.\, A}{a \sin am} = \frac{b \sin b \,.\, B}{b \sin bm} = \frac{c \sin c \,.\, C}{c \sin cm} = \frac{\Sigma A^2 (B-C)}{\Sigma a \sin am \,.\, A} =$$

$$-\frac{(B-C)(C-A)(A-B)}{ab \sin C \sin tm} = -2R \frac{\sin a \sin b \sin c}{\sin tm}.$$

Collecting our results, we have $\Sigma a \sin am \,.\, BC = 0$; $\Sigma a \sin am \,.\, X^2 = 0$, and

$$\frac{\sin a \,.\, A}{\sin am} = \frac{\sin b \,.\, B}{\sin bm} = \frac{\sin c \,.\, C}{\sin cm} = -2R \frac{\sin a \sin b \sin c}{\sin tm},$$

eight equations to the same parabola. One look at them is sufficient to show that by the orthogonal substitution of the tangent at the vertex for the axis (or m), we obtain the equation to its envelope. For, in this case, sin am, sin bm, sin cm, become cos a, cos b, cos c, and sin $tm = 1$. Hence $\Sigma a \cos a \,.\, BC = 0$; $\Sigma a \cos a \,.\, X^2 = 0$; and tan $a \,.\, A = \tan b \,.\, B = \tan c \,.\, C = -2R \sin a \sin b \sin c$, must be the equations to the envelope of the tangents at the vertices of the parabolas inscribed in the triangle of reference. The curve is now well known to the readers of this Journal to be the tricusped hypocycloid, or tripod, as I shall venture to call it.

The last equations bring the tripod to a certain extent within the domain of Goniometry, so that it can be shown that whilst all these equations are cases of ease of one identity, another is intimately connected with the five remarkable points on a tangent.

In the rectangular tangent (A′B′C′), $\cot a \,.\, A' = \cot b \,.\, B' = \cot c \,.\, C'$. Hence $AA' = BB' = CC'$, and the curve is also the envelope of the rectangular asymptotes of conics circumscribing the triangle.

The equation $A \sin tm = -2R \sin am \sin b \sin c$ is curious. It contains, in fact, two sets of variables. Thus $2R \sin am = AS$ (S the focus) is an equation to the parabola. What is its meaning, when A is on the curve? How can the tangential distances of A from the curve be at once found? Why is the limit of (sin $tm \,.\, A$), t being the tangent at infinity, constant

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whatever is the position of A? From this equation, and $ef \sin b \sin c = \sin A \cdot A$, we get $ef \sin tm = a \sin am$, and for the vertical tangent this becomes $ef = a \cos a$.

Thus the tripod which first picks out from the parabolas the tangents of least intercept, joins to any two of them the tangent of greatest intercept between them to form a generating triangle.

I leave it for the reader to remark the constant relations connecting the squares of the intercepts of rectangular tangents between any two tangents, first to the parabola and then to the tripod.

1684. (Proposed by J. CASEY, B.A.)—Prove by inversion the leading properties of the Lemniscate of Bernoulli, including those of its osculating circle and Saladini's property that it is the curve down which a particle will descend through any arc in the same time as it will take to descend through the chord of the arc. Show also that it possesses, in a modified way, anharmonic properties similar to those of the conic sections.

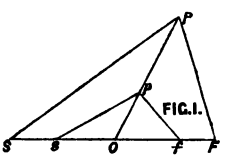
Solution by the PROPOSER.

1. Let Ox, Oy be the asymptotes of an equilateral hyperbola, PY a tangent at any point P, and OY a perpendicular on it from O; and let OY be produced to meet the hyperbola in Q; then it is easy to see that OP, OY are equally inclined to the axis, and that $OY \cdot OP = OY \cdot OQ = a^2$ (a being the semi-axis); therefore Q and Y are *inverse points* with respect to the circle whose radius is a and centre O. Now the central polar equation of the hyperbola being $\rho^2 \cos 2\theta = a^2$, that of the locus of Y is $a^2 \cos 2\theta = \rho^2$; hence this locus is the *Lemniscate of Bernoulli*; and corresponding to the asymptotes of the hyperbola are the inflectional tangents of the Lemniscate; in fact, they are the same lines.

2. Let s, f, (Fig. 1) be the inverse points of the foci S, F, and p the inverse of P; then by similar triangles, we have

$SP : PO = sp : sO$, $FP : PO = fp : fO$;

but $SP \cdot FP = PO^2$, hence $sp \cdot fp = so \cdot fo$; hence the lemniscate is the locus of the vertex of a triangle, whose base is given, and the rectangle under whose sides is equal to the square on half the base.



3. The rectangle contained by the intercepts made by a tangent on the asymptotes of the hyperbola is constant. Hence *if a circle pass through the origin and touch the lemniscate, the rectangle contained by the intercepts which it makes on the inflectional tangents is constant.*

4. The tangent and normal to a hyperbola bisect the angles between the focal radii vectores. Hence, (see Townsend's *Modern Geometry*, Vol. II., Chapter on Inversion,) *if through the origin and any point* P *in the lemniscate two circles be described passing through its two foci, of their two circles of inversion, one touches the curve at* P *and the other cuts it orthogonally at* P.

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5. If any line intersect the hyperbola and its asymptotes, the intercepts between the hyperbola and asymptotes are equal. Hence, *if a circle passing through the origin* O *intersect the lemniscate in the points b, c, and the inflectional tangents in the points a and d, we have* $ab \cdot bd : ac \cdot cd = Ob^2 : Oc^2$.

6. The parallels to the tangent at the vertex of any diameter are bisected by that diameter. Hence, *if a series of circles touch each other at the centre of the lemniscate, the radii vectores drawn to it from the points where they cut the curve again form a pencil in involution, the double lines of which are the common tangent to the circles and the line drawn to the point of contact of that circle of the system which touches the lemniscate.*

7. If XPY be a tangent to the hyperbola, intersecting the asymptotes in X and Y, it is evident that $\angle XPO = 2POY$. Hence *if* OP′ *be a radius vector to a lemniscate, and* P′Y′ *a tangent at* P′ *intersecting the inflectional tangent in* Y′, *then* $\angle OP'Y' = 2\,P'OY'$.

8. If B, C be two consecutive points in the hyperbola, and through B, C, perpendiculars be drawn to one of the asymptotes; then if OB be joined and produced to meet the parallel through C in D, it is easily seen that BC = BD. Hence if OB′X, OC′X′ (Fig. 2) be two semicircles through two consecutive points B′C′, and if OB′, OC′ be joined and OB′ be produced to meet the semicircle OC′X′ in D′, the element B′C′ = B′D′. If therefore OX be vertical, and two material particles move under the action of gravity along OB′, OC′; and if the particle which moves along OB′, when it arrives at B′, move with the same velocity along B′C′; the particles will both arrive at C′ at the same instant. From this it readily follows that a material particle will arrive at C′ in the same time whether it moves along the curve or chord. This is Saladini's beautiful property of the Lemniscate.

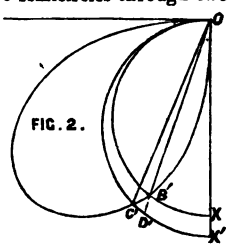

FIG. 2.

9. If PVQ be the circle of curvature at any point P, and if PV be the chord of curvature, then PV = 2OP. Hence *if* P′V′Q′ *be the osculating circle at any point* P′ *of the Lemniscate, it divides the radius vector* OP′ *in the ratio of* 1 : 2.

10. Let OR (Fig. 3) be a radius vector to the hyperbola, RP′ the tangent at R, OP′ the perpendicular on RP′, P′X the tangent to the lemniscate at P′, and OX the inflectional tangent. Now if P′S be perpendicular to P′X, we have by Art. 7, $\angle OP'X = 2P'OX$, whence it follows that $\angle SP'O = SOP$; therefore OR is bisected in S, and the circle of curvature cuts P′O in V′ making $P'V' = \frac{2}{3}OP'$; hence it is seen at once that the centre of curvature at P′ is the centre of gravity of the triangle OP′R, and that the radius of curvature is $\frac{1}{3}$OR.

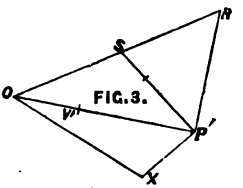

FIG. 3.

11. Hence the length of the evolute of the lemniscate between two points whose radii vectores are ρ_1, ρ_2 is $\frac{1}{3}a^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$; for it $= \frac{1}{3}OR_1 - \frac{1}{3}OR_2$.

12. Since the perpendiculars at the extremities of the radii vectores of the lemniscate are tangents to the hyperbola, it follows that any two are cut homographically by all the rest.

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13. If Q be a variable point, and A, B, C, D, four fixed points on the hyperbola, the pencil $\{Q \,.\, ABCD\}$ is constant. Hence *if* Q′ *be a variable and* A′B′C′D′ *four fixed points in the lemniscate, the four circles* OQ′A′, OQ′B′, OQ′C′, OQ′D′, *intersect at a constant anharmonic ratio, and since the circles are coaxial, their centres are in a right line, and the anharmonic ratio of the four centres is given.*

It would be easy to multiply these properties, but enough has been given to show the great power of the method of inversion, and its utility in the Higher Geometry.

1766. (Proposed by J. Wilson.)—To find the condition necessary in order that one of two homofocal ellipses should be inscribed in a triangle, and the other circumscribed about it.

Solution by Professor Cremona.

Soient deux coniques rapportées au triangle conjugué commun

$$ax^2 + by^2 + cz^2 = 0, \qquad a'x^2 + b'y^2 + c'z^2 = 0.$$

Si la première est circonscrite à un triangle dans lequel le deuxième soit inscrite, tout point de la première est un sommet d'un triangle doué de la même propriété. Du point $(x = 0,\ y : z = i\sqrt{c} : \sqrt{b})$ de la première conique menons à l'autre deux tangentes qui seront représentées par l'équation

$$(bc' - b'c)\,(a'x^2 + b'y^2 + c'z^2) - (ib'y\sqrt{c} + c'z\sqrt{b})^2 = 0,$$

et couperont la première conique en deux points situés sur la droite

$$i\sqrt{b}(ab'c' - a'bc' + a'b'c)y + \sqrt{c}(ab'c' + a'bc' - a'b'c)z = 0.$$

Cette droite est tangente à la seconde conique si l'on satisfait à la condition

$$bc'(ab'c' - a'bc' + a'b'c)^2 - b'c(ab'c' + a'bc' - a'b'c)^2 = 0,$$

c'est-à-dire $a^2b'^2c'^2 + b^2c'^2a'^2 + c^2a'^2b'^2 - 2a'b'c'\,(bca' + cab' + abc') = 0.$

C'est la condition pour que la première conique passe par les sommets d'un triangle circonscrit à la seconde.

Si les coniques proposées sont deux ellipses homofocales

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - e^2} - 1 = 0, \qquad \frac{x^2}{a'^2} + \frac{y^2}{a'^2 - e^2} - 1 = 0,$$

il suffira de prendre

$$a : b : c = a^2 - e^2 : a^2 : -a^2\,(a^2 - e^2),$$
$$a' : b' : c' = a'^2 - e^2 : a'^2 : -a'^2(a'^2 - e^2);$$

et l'on aura pour la condition demandée

$$a^8 - 4a^6a'^2 + 6a^4a'^2e^2 - 4a^2a'^2e^4 + a'^4e^4 = 0.$$

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1737. (Proposed by Dr. BOOTH, F.R.S.)—Solve the equations

$$axy + a'xz + a = 0, \quad \beta yz + \beta' yx + b = 0, \quad \gamma zx + \gamma' zy + c = 0.$$

Solution by J. E. RADCLIFFE.

Putting $x^{-1} = \xi$, $y^{-1} = v$, $z^{-1} = \zeta$, $\xi v \zeta = \phi$, the equations become

$$a\zeta + a'v + a\phi = 0 \ldots (1), \quad \beta\xi + \beta'\zeta + b\phi = 0 \ldots (2), \quad \gamma v + \gamma'\xi + c\phi = 0 \ldots (3).$$

Then from P (1) + Q (2) + R (3) we have

$$(Pa + Q\beta')\,\zeta + (Q\beta + R\gamma')\,\xi + (Pa' + R\gamma)\,v + (Pa + Qb + Rc)\,\phi = 0.$$

Now to obtain the value say of ζ, let the coefficients of ξ and v be each put $= 0$, or $Q = -R\gamma/\beta^{-1}$, $P = -R\gamma a'^{-1}$; then, putting $a\beta\gamma + a'\beta'\gamma' = \Delta$, and $ca'\beta - a\beta\gamma - ba'\gamma' = C$, &c., we have

$$\zeta = \frac{C\phi}{\Delta}, \text{ in like manner } \xi = \frac{A\phi}{\Delta}, \text{ and } v = \frac{B\phi}{\Delta};$$

$$\therefore \xi v \zeta = \phi = \frac{ABC\phi^3}{\Delta^3}, \text{ or } \phi = \frac{\Delta^{\frac{3}{2}}}{(ABC)^{\frac{1}{2}}}; \quad \therefore \xi = \frac{A\phi}{\Delta} = \left(\frac{A\Delta}{BC}\right)^{\frac{1}{2}};$$

$$\text{hence } x = \left(\frac{BC}{A\Delta}\right)^{\frac{1}{2}}, \; y = \left(\frac{CA}{B\Delta}\right)^{\frac{1}{2}}, \; z = \left(\frac{AB}{C\Delta}\right)^{\frac{1}{2}}.$$

Solution by ABRACADABRA; S. BILLS; J DALE; *and others.*

We may write the equations thus:—

$$\left.\begin{array}{rrrr} a\,.\,xy + & a'\,.\,xz + & & + a = 0 \\ \beta'\,.\,xy + & & + \beta\,.\,yz + & b = 0 \\ & \gamma\,.\,xz + & \gamma'\,.\,yz + & c = 0 \end{array}\right\};$$

$$\therefore \frac{xy}{\Sigma \pm (a', \beta, c)} = \frac{xz}{\Sigma \pm (0, b, 0)} = \frac{yz}{\Sigma \pm (a, \beta', \gamma')} = -\frac{1}{a\beta\gamma + a'\beta'\gamma'}$$

$$\therefore x = \left\{-\frac{[\Sigma \pm (a', \beta, c)] \times [\Sigma \pm (0, b, 0)]}{[\Sigma \pm (a, \beta', \gamma)] \times (a\beta\gamma + a'\beta'\gamma)}\right\}^{\frac{1}{2}}$$

$$y = \left\{-\frac{[\Sigma \pm (a, \beta', \gamma)] \times [\Sigma \pm (a', \beta, c)]}{[\Sigma \pm (0, b, 0)] \times (a\beta\gamma + a'\beta'\gamma')}\right\}^{\frac{1}{2}}$$

$$z = \left\{-\frac{[\Sigma \pm (0, b, 0)] \times [\Sigma \pm (a, \beta', \gamma)]}{[\Sigma \pm (a', \beta, c)] \times (a\beta\gamma + a'\beta'\gamma')}\right\}^{\frac{1}{2}}.$$

1712. (Proposed by the Rev. R. HARLEY, F.R.S.)—Let $f(x)$ be any rational integral function of x, of the degree n, and $f'(x)$ the first derived

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function; then will $f'(x_1) \cdot f'(x_2) \ldots f'(x_n) = a^{2-n} n^n \Delta$, where a is the coefficient of the highest power of x in $f(x)$, Δ is the discriminant, and $x_1, x_2 \ldots x_n$ are the roots of $f(x) = 0$.

Solution by the PROPOSER.

Since, identically,

$$f(x) = a(x-x_1)(x-x_2) \ldots\ldots\ldots (x-x_n),$$

therefore, by differentiation, we have

$$\begin{aligned} f'(x) &= a(x-x_2)(x-x_3)\ldots\ldots\ldots(x-x_n) \\ &\quad + a(x-x_1)(x-x_3)\ldots\ldots\ldots(x-x_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + a(x-x_1)(x-x_2)\ldots\ldots(x-x_{n-1}), \end{aligned}$$

$$f'(x_1) = a(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n),$$

$$f'(x_2) = a(x_2-x_1)(x_2-x_3)\ldots\ldots\ldots(x_2-x_n),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f'(x_n) = a(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1});$$

therefore $\quad f'(x_1) \cdot f'(x_2) \ldots f'(x_n) = (-)^{\frac{1}{2}n(n-1)} a^n \mathrm{S},$

where, for shortness, S is written in place of

$$(x_1-x_2)^2 (x_1-x_3)^2 \ldots (x_2-x_3)^2 (x_2-x_4)^2 \ldots$$

the product of the squares of the differences of the roots.

Now if Δ be expressed as a function of the coefficients of the equation $f(x) = 0$, and if l be the last or absolute term in this equation, we know that the expression for Δ will contain the term $+a^{n-1} l^{n-1}$; and since Δ is to a factor *près* equal to S, we may write $a^{2n-2}\,\mathrm{S} = k\Delta$, where k is a number. To determine this number, let us use the binomial equation

$$x^n - 1 = 0, \text{ for which } a^n \mathrm{S} = (-)^{n-1} k,$$

and therefore $\quad f'(x_1) \cdot f'(x_2) \ldots f'(x_n) = (-)^{\frac{1}{2}n(n-1)+(n-1)} k.$

But in this case

$$f(x) = x^n - 1, \quad f'(x) = nx^{n-1}, \quad f'(x_1) = nx_1^{n-1}, \text{ \&c.};$$

therefore $\quad f'(x_1) \cdot f'(x_2) \ldots f'(x_n) = n^n (x_1 x_2 \ldots x_n)^{n-1};$

or, since $\quad x_1 x_2 \ldots x_n = (-1)^{n-1}, \quad$ we have

$$f'(x_1) \cdot f'(x_2) \ldots f'(x_n) = (-)^{(n-1)^2} n^n = (-)^{n-1} n^n;$$

$$\therefore (-)^{\frac{1}{2}n(n-1)+(n-1)} k = (-)^{n-1} n^n; \qquad \text{whence } k = (-)^{\frac{1}{2}n(n-1)} n^n;$$

so that, in general,

$$a^{2n-2}\,\mathrm{S} = (-)^{\frac{1}{2}n(n-1)} n^n \Delta, \qquad \text{or } \mathrm{S} = (-)^{\frac{1}{2}n(n-1)} a^{-2n+2} n^n \Delta;$$

$$\therefore f'(x_1) \cdot f'(x_2) \ldots f'(x_n) = (-)^{n(n-1)} a^{-n+2} n^n \Delta = a^{-n+2} n^n \Delta.$$

Since I proposed this question for solution, I have discovered that Prof. CAYLEY, in article 82 of his Fourth Memoir upon Quantics (*Phil. Trans.*

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for 1858,) has determined the factor here denoted by k. I may mention, however, that the value of the discriminant given in that article requires to be slightly amended by writing a^{2m-2} in place of a^{m-2}, and that the same correction should be made in the last line on page 423.

The property established above is important on account of its connection with the theory of differential resolvents.

1724. (Proposed by W. K. CLIFFORD.)—The equations of three conics being given in the following forms: viz.,

$$a_1x^2+b_1y^2+c_1z^2+d_1w^2=0, \quad a_2x^2+\&c.=0, \quad a_3x^2+\&c.=0,$$

where $x+y+z+w \equiv 0$, show that a straight line $(\xi x+\eta y+\zeta z+\omega w=0)$ will be cut in involution by them, if

$$\Sigma . \begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix} . (\xi-\eta)(\xi-\zeta)(\xi-\omega) . (\text{to four terms})=0.$$

Solution by W. S. BURNSIDE, B.A.

The condition that the lines given by the equations

$$Lx^2+2Mxy+Ny^2=0, \quad L_1x^2+2M_1xy+N_1y^2=0, \quad L_2x^2+2M_2xy+N_2y^2=0,$$

should form a pencil in involution is

$$\begin{vmatrix} L & M & N \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix} = 0, \text{ or } (LM_1N_2)=0 \ldots\ldots (I.)$$

To apply this condition to the present Question, eliminate y and z from the equations of the conics by means of the equations $x+y+z+w=0$ and $x\xi+y\eta+z\zeta+w\omega=0$, and we shall find the equations of the lines joining (x, y) to the intersections of each conic with the line $x\xi+y\eta+z\zeta+w\omega=0$.

For shortness, let $\xi-\omega=\alpha, \quad \eta-\omega=\beta, \quad \zeta-\omega=\gamma$;

whence $x\xi+y\eta+z\zeta+w\omega=0$ becomes $ax+\beta y+\gamma z=0$;

also let $(b_1c_2d_3)=A, (c_1d_2a_3)=B, (d_1a_2c_3)=C, (a_1b_2c_3)=D$;

then the condition (I.) is found to be

$$\begin{vmatrix} a_1\gamma^2+c_1\alpha^2+d_1(\gamma-\alpha)^2, & b_1\gamma^2+c_1\beta^2+d_1(\gamma-\beta)^2, & c_1\alpha\beta+d_1(\gamma-\alpha)(\gamma-\beta) \\ a_2\gamma^2+c_2\alpha^2+d_2(\gamma-\alpha)^2, & b_2\gamma^2+c_2\beta^2+d_2(\gamma-\beta)^2, & c_2\alpha\beta+d_2(\gamma-\alpha)(\gamma-\beta) \\ a_3\gamma^2+c_3\alpha^2+d_3(\gamma-\alpha)^2, & b_3\gamma^2+c_3\beta^2+d_3(\gamma-\beta)^2, & c_3\alpha\beta+d_3(\gamma-\alpha)(\gamma-\beta) \end{vmatrix}.$$

There are 18 determinants here, 12 of which vanish, and the remaining 6 are (when γ^2 is divided off, which was introduced before)

$$D\gamma^2\alpha\beta+C\gamma^2(\gamma-\alpha)(\gamma-\beta)+B\beta\alpha(\gamma-\beta)^2-B\beta^2(\gamma-\alpha)(\gamma-\beta)+$$
$$A\alpha\beta(\gamma-\alpha)^2-A\alpha^2(\gamma-\alpha)(\gamma-\beta)=0,$$

or $A\alpha(\alpha-\beta)(\alpha-\gamma)+B\beta(\beta-\gamma)(\beta-\alpha)+C\gamma(\gamma-\alpha)(\gamma-\beta)+D\alpha\beta\gamma=0,$

which is equivalent to the condition given in the Question.

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1497. (Proposed by W. K. CLIFFORD.)—(1.) Given three points by equations of the form $lx + my + nz = 0$; prove that the area of the triangle contained by them is

$$(l_1m_2n_3) + (l_1 + m_1 + n_1)(l_2 + m_2 + n_2)(l_3 + m_3 + n_3),$$

that of the triangle of reference being unity.

(2.) Also, if (123) denote the area of the triangle contained by the points 1, 2, 3, and so on, prove that

$$(123)(456) \equiv (156)(423) + (164)(523) + (145)(623).$$

Solution by the PROPOSER.

1. The area of the triangle formed by three points vanishes only when they are in a straight line, and becomes infinite only when one of them is at infinity. The condition that they may be in a straight line is

$$J \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0, \text{ or } (l_1m_2n_3) = 0;$$

and the condition that one of them may be at infinity is

$$P \equiv (l_1 + m_1 + n_1)(l_2 + m_2 + n_2)(l_3 + m_3 + n_3) = 0.$$

Now the expression for the area must be of no dimensions in the coefficients, since we are only concerned with their ratios; and the equation obtained by equating this expression to a constant must be of the first order in each set of coefficients; since, two of the points being fixed, the locus of the other is then a straight line. The expression for the area is therefore some numerical multiple of $\frac{J}{P}$. By putting $x,y,z = 0$ for the three points, we find that the area of the fundamental triangle, on the same scale, is unity.

2. To prove the second proposition, take (456) for the fundamental triangle. Then, by applying the above interpretation to the well-known theorem

$$(l_1m_2n_3) \equiv l_1(m_2n_3) + m_1(n_2l_3) + n_1(l_2m_3)$$

we find it equivalent to

$$(123)(456) \equiv (156)(423) + (164)(523) + (145)(623),$$

the factor P dividing out on both sides.

1733. (Proposed by W. A. WHITWORTH, M.A.)—To find the area of a triangle the equations of whose sides in trilinear coordinates are

$$l_1\alpha + m_1\beta + n_1\gamma = 0, \quad l_2\alpha + m_2\beta + n_2\gamma = 0, \quad l_3\alpha + m_3\beta + n_3\gamma = 0.$$

Solution by ABRACADABRA.

Let Δ' denote the required area, Δ the area of the triangle of reference, then (Salmon's Conics, 4th ed., Art. 39), we shall have

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$$2\Delta' = \frac{\left\{\Sigma \pm (l_1 \cos\alpha + \&c.,\ l_2 \sin\alpha + \&c.),\ -l_3 p_\alpha - m_3 p_\beta - \&c.\right\}^2}{\left\{\Sigma \pm (l_1 \cos\alpha + \&c.,\ l_2 \sin\alpha + \&c.)\right\}\left\{\Sigma \pm (l_2 \cos\alpha + \&c.,\ \&c.)\right\}\left\{\Sigma \pm (\&c.,\ \&c.)\right\}}$$

$$\left(\text{or, putting } V = \frac{\sin(\beta-\gamma)}{a} = \frac{\sin(\gamma-\alpha)}{b} = \frac{\sin(\alpha-\beta)}{c}\right)$$

$$= \frac{\left\{a \cdot p_\alpha + b \cdot p_\beta + c \cdot p_\gamma\right\}^2 \cdot \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}^2}{V\left\{c \cdot \begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} + a \cdot \begin{vmatrix} m_1, & n_1 \\ m_2, & n_2 \end{vmatrix} + b \cdot \begin{vmatrix} n_1, & l_1 \\ n_2, & l_2 \end{vmatrix}\right\}\left\{c \cdot \begin{vmatrix} l_2, & m_2 \\ l_3, & m_3 \end{vmatrix} + \&c.\right\}\left\{c \cdot \begin{vmatrix} l_3, & m_3 \\ l_1, & m_1 \end{vmatrix} + \&c.\right\}};$$

and hence twice the area of the triangle of reference is

$$2\Delta = \frac{\left\{a \cdot p_\alpha + b \cdot p_\beta + c \cdot p_\gamma\right\}^2 l_1^2 m_2^2 n_3^2}{V \cdot abc \cdot l_1^2 m_2^2 n_3^2}$$

$$\therefore \frac{\Delta'}{\Delta} = \frac{abc \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}^2}{\begin{vmatrix} a, & b, & c \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} \cdot \begin{vmatrix} a, & b, & c \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \cdot \begin{vmatrix} a, & b, & c \\ l_3, & m_3, & n_3 \\ l_1, & m_1, & n_1 \end{vmatrix}}.$$

II. *Solution by the* PROPOSER; M. JENKINS, B.A.; *and others.*

Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the vertices of the triangle Δ', expressed by trilinear coordinates; then in the Solution of Question 1497 (*Reprint*, Vol. I., p. 80), it is shown that

$$\Delta' = \frac{abc}{8\Delta^2}, \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(I.)}$$

But x_1, y_1, z_1 are obtained by solving the simultaneous system

$$l_2 x_1 + m_2 y_1 + n_2 z_1 = 0, \quad l_3 x_1 + m_3 y_1 + n_3 z_1 = 0, \quad a x_1 + b y_1 + c z_1 = 2\Delta;$$

and are therefore given by

$$\frac{x_1}{\begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}} = \frac{y_1}{\begin{vmatrix} n_2 & l_2 \\ n_3 & l_3 \end{vmatrix}} = \frac{z_1}{\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix}} = \frac{2\Delta}{\begin{vmatrix} a & b & c \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}}$$

and x_2, y_2, z_2; x_3, y_3, z_3 are given by similar equations; hence we have

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$$\Delta' = \frac{abc\,\Delta \begin{vmatrix} \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}, & \begin{vmatrix} n_2 & l_2 \\ n_3 & l_3 \end{vmatrix}, & \begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} \\ \begin{vmatrix} m_3 & n_3 \\ m_1 & n_1 \end{vmatrix}, & \begin{vmatrix} n_3 & l_3 \\ n_1 & l_1 \end{vmatrix}, & \begin{vmatrix} l_3 & m_3 \\ l_1 & m_1 \end{vmatrix} \\ \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}, & \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, & \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \cdot \begin{vmatrix} a & b & c \\ l_3 & m_3 & n_3 \\ l_1 & m_1 & n_1 \end{vmatrix} \cdot \begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}} \quad \ldots\ldots (II.)$$

But the determinant forming the numerator of the last fraction is evidently equivalent to

$$\frac{1}{l_1} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \cdot \begin{vmatrix} \begin{vmatrix} n_3 & l_3 \\ n_1 & l_1 \end{vmatrix}, & \begin{vmatrix} l_3 & m_3 \\ l_1 & m_1 \end{vmatrix} \\ \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}, & \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \end{vmatrix}$$

And we find that

$$\begin{vmatrix} \begin{vmatrix} n_3 & l_3 \\ n_1 & l_1 \end{vmatrix} & \begin{vmatrix} l_3 & m_3 \\ l_1 & m_1 \end{vmatrix} \\ \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} & \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \end{vmatrix} \equiv l_1 \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

Hence the expression (II.) for the area (Δ') required becomes

$$\Delta' = abc\,\Delta \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 \div \left\{ \begin{vmatrix} a & b & c \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \cdot \text{ \&c. \&c.} \right\} \quad \ldots\ldots (III.).$$

III. *Solution by* W. K. Clifford.

Call the three lines 1, 2, 3. Then we have to find the area of the triangle included by the points (23), (31), (12); that is, by the points

$$L_1x + M_1y + N_1z = 0, \quad L_2x + M_2y + N_2z = 0, \quad L_3x + M_3y + N_3z = 0;$$

where $L_1 \ldots$ are the first minors of the determinant $(l_1m_2n_3)$. But by the Solution to Quest. 1497, this area is a numerical multiple of $\frac{J}{P}$, where

$$J \equiv (L_1M_2N_3) \equiv (l_1m_2n_3)^2$$

$$P \equiv (aL_1 + bM_1 + cN_1)\,(aL_2 + bM_2 + cN_2)\,(aL_3 + bM_3 + cN_3)$$

in the Trilinear system.

In the case of the fundamental triangle we find that $\frac{J}{P} = \frac{1}{abc}$. Hence the *ratio* has the value given in the foregoing Solutions.

The general formula for all systems (see my paper on "Analytical Metrics" in the *Quarterly Journal of Mathematics*, Vol. vii. p. 62) is

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$$\frac{\{J(ABC)\}^2}{J(BC\infty)\,.\,J(CA\infty)\,.\,J(AB\infty)}$$

where $J(ABC)=0$ is the condition that the lines A, B, C, $=0$ may meet in a point, and $J(BC\infty)=0$ is the condition that B and C may be parallel. The *ratio* to the area of the fundamental triangle may easily be found in any particular case by the method used above.

In the same way it may be shown that the volume of a tetrahedron is

$$\frac{\{J(ABCD)\}^3}{J(BCD\infty)\,.\,J(CDA\infty)\,.\,J(DAB\infty)\,.\,J(ABC\infty)}$$

where $J(ABCD)=0$ is the condition that the planes A, B, C, D$=0$ may meet in a point, and $J(BCD\infty)=0$ is the condition that B, C, D may be parallel to the same line.

In quadriplanar coordinates, for instance, if α, β, γ, δ, denote the areas of the faces of the fundamental tetrahedron, the equation to the plane at infinity is

$$\alpha x+\beta y+\gamma z+\delta w=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the above expression for the volume, if calculated by means of (1), must be multiplied by the product $\alpha\beta\gamma\delta$ to give the ratio of the volume of the given tetrahedron to that of the fundamental one.

1732. (Proposed by T. A. Hirst, F.R.S.)—Prove that the characteristics (see Question 1573) of a system of conics, satisfying four conditions, remain unaltered when, in place of passing through a given point, each conic is required to divide a given finite segment harmonically.

Solution by W. K. Clifford.

In a system of conics satisfying four conditions (Z, Z', Z'', Z''') let μ be the number of conics that pass through an arbitrary point, and ν the number that touch an arbitrary line. Suppose that the polars of a point P, in respect of all the conics of the system, envelope a curve of class x. Then from the point P, x tangents can be drawn to the curve; that is to say, there are x polars of P which pass through P. But a point which lies on its polar in respect of a given conic is a point on the conic. Therefore x conics pass through P. But μ conics (by hypothesis) pass through P; so that $x=\mu$. Thus we get Chasles' Prop. XII.,—*the polars of an arbitrary point, in respect of a system of conics* (μ, ν) *envelope a curve of class* μ. It follows that there are μ polars of P which pass through another arbitrary point Q; that is to say, there are μ conics of the system which divide harmonically a given segment PQ. This is Chasles' Prop. XXVIII.

Suppose now that the condition Z''' is that the conics shall pass through a given point. Call S the condition that they shall divide harmonically the segment PQ. Then (by the above) the number of conics satisfying the conditions (Z, Z', Z'', S), and passing through the given point, is μ; that is to say, the *first characteristic* (μ') of the system (Z, Z', Z'', S) is equal to the *first characteristic* (μ) of the system (Z, Z', Z'', Z''') where Z''' is the

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condition of passing through a given point. In the next place let Z''' be the condition of touching a given line. Then the number of conics which satisfy the conditions (Z, Z', Z'', Z''', S) is the same as the number which satisfy the conditions (Z, Z', Z'', Z''', point); that is to say, the *second characteristic* (ν') of the system (Z, Z', Z'', S) is equal to the *second characteristic* (ν) of the system (Z, Z', Z'', point). Thus neither of the characteristics is altered when we substitute for the condition of passing through a given point, the condition S of dividing harmonically a given segment PQ.

By similar reasoning it may be shown that neither characteristic is altered when we substitute for the condition of touching a given line, the condition of subtending harmonically a given angle.

1709. (Proposed by H. McColl.)—1. Let $f(x)$ be any function of x, and let $f(a, b)$ express (according as a is greater or less than b) a superior or inferior limit to $f(x)$ for all values of x between a and b; show that if $f(a, b)$ has the same sign as $b-a$, no real root of the equation $f(x)=0$ exists between a and b.

2. Let $f(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$, n being a positive integer. Show that, when a and b are positive, $f(a, b)$ may be found by the following process. According as A_n is positive or negative, multiply it by a or b; add the product to A_{n-1}, and represent the result by k_1. According as k_1 is positive or negative, multiply it by a or b; add the product to A_{n-2} and represent the result by k_2. Proceeding in this way till $k_1, k_2, k_3, \ldots$ k_n are all found, we shall have $f(a, b) = k_n$.

Solution by the Proposer.

1. First, let $f(a, b)$ and $b-a$ be both positive. In this case, b being algebraically greater than a, the symbol $f(a, b)$ expresses an *inferior* limit to $f(x)$ for all values of x between a and b; so that, when x is thus restricted, $f(x)$ must always be greater than the positive quantity $f(a, b)$, and cannot, therefore, be equal to zero. Next, let $f(a, b)$ and $b-a$ be both negative. In this case, b being less than a, the symbol $f(a, b)$ expresses a *superior* limit to $f(x)$ for all values of x between a and b; so that when x is thus restricted, $f(x)$ must always be less (algebraically) than the negative quantity $f(a, b)$, and cannot, therefore, be equal to zero. The terms *superior* and *inferior* are, of course, used in their algebraic and not their numerical sense.

2. Let x_1 denote any value of x between a and b. Multiply A_n by x_1; add the product to A_{n-1}; and represent the result by r_1. Multiply r_1 by x_1; add the product to A_{n-2}; and represent the result by r_2. Proceeding thus by Horner's method, we shall ultimately have $r_n = f(x_1)$. First, let $f(a, b)$ express a superior limit to $f(x)$ for all values of x between a and b. In this case, a is greater than b; and it requires no formal demonstration to show that k_1 is greater (algebraically) than r_1; that k_2 is greater than r_2; and so on to k_n, which is greater than r_n or $f(x_1)$. Next, let $f(a, b)$ express an

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inferior limit to $f(x)$ for all values of x between a and b. In this case, a is less than b; and it may be proved similarly that k_1 is algebraically less than r_1; that k_2 is less than r_2; and so on to k_n, which is less than r_n or $f(x_1)$.

N.B.—When $f(a)$ and $f(b)$ are positive, we must, in finding $f(a, b)$, make a represent the less quantity; but when $f(a)$ and $f(b)$ are negative, we must make a represent the greater quantity; otherwise, $f(a, b)$, when obtained, will not give the desired information.

This method will frequently detect impossible roots by inspection. Take, for example, the equation $2x^4 - x^3 + 5x^2 - 3x + 4 = 0$. We find that $f(0, 1)$ is positive, which shows that no real root exists between 0 and 1; and next we find that $f(1, \infty)$ is positive, which shows that no real root exists between 1 and ∞.

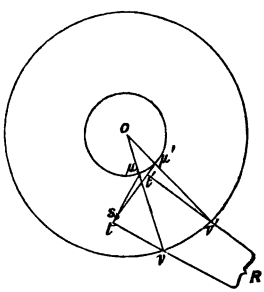
$2-1+5-3+4$ (0, 1	$2-1+5-3+4$ (1, ∞
$0-1+0-3$	$+2+1+6+3$
$-1+4-3+1$	$+1+6+3+7$

On the regular Tricusped Hypocycloid.

By M. Jenkins, B.A.

1. *Evolute.*—The evolute of the regular tricusped hypocycloid is another regular tricusped hypocycloid, and of *any* hypocycloid is a *similar* hypocycloid, lying between two circles (which may be called the pericentric and apocentric circles, since all hypocycloids are particular cases of planetary motion) such that the pericentric circle of the evolute coincides with the apocentric circle of the original hypocycloid, and the apse points of the evolute coincide with the cusp-points of the original hypocycloid.

It follows immediately from the ordinary definition of a hypocycloid, that the angle between two consecutive tangents has a constant ratio (κ, suppose) to the angle subtended by the intersections (μ, μ') of the tangents with the pericentric circle at the circumference of that circle; μ, μ' being the points of intersection with the circle nearest to s, the intersection of the tangents. This ratio in the case of the regular tricusp is one of equality (that is, $\kappa = 1$).



Let μt, $\mu' t'$ be two consecutive tangents cutting the pericentric circle in μ, μ', and each other in the point s. Draw $O\mu$, $O\mu'$ through the centre of the circle, meeting the apocentric circle in ν, ν'; and draw νt, $\nu' t'$ perpendicular to μt, $\mu' t'$ respectively. Then t, t' are the points of contact of the tangents μt, $\mu' t'$.

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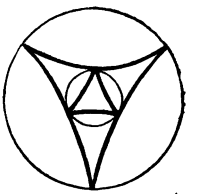
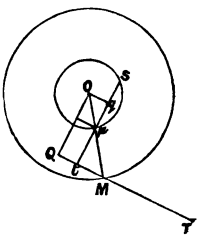


Let νt, $\nu' t'$ meet in R; then the locus of R is the evolute. Now νR, ν'R are consecutive tangents to the evolute, and νt, $\nu' t'$ are perpendicular to μt, $\mu' t'$ respectively; hence, for *any* hypocycloid

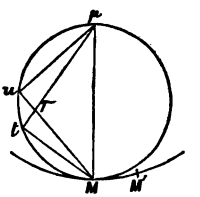
$$\angle \nu R \nu' = \mu s \mu' = \kappa \cdot \tfrac{1}{3} \mu O \mu' = \kappa \cdot \tfrac{1}{3} \nu O \nu'$$
$$= \kappa \cdot \angle \text{ subtended at circumference by } \nu\nu';$$

therefore the locus of R is a *similar* hypocycloid having the apocentric circle of the original hypocycloid for its pericentric circle. And since a tangent to the evolute is a normal to the original hypocycloid, the apse points of the evolute are on the cusp-points of the original hypocycloid.

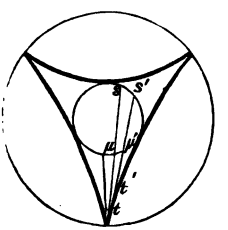


2. *Radius of Curvature.*—Let μt be a tangent, μ its centre, s its vertex, and t the point of contact; O the centre of the pericentric and apocentric circles; and M the point where $O\mu$ cuts the apocentric circle. Then Mt is perpendicular to μt. Let OQ, parallel to μt, meet Mt in Q, make QT = 3QM, and draw Oq perpendicular to μs.

Then, since $\mu q = \frac{1}{3}\mu s = \frac{1}{3}\mu t$, $qt = 3\mu q$; and since tT is a tangent to the evolute, and QT = 3QM, therefore T is the point of contact of MT with the evolute; and QT = 3QM = 9Oq; hence tT (the radius of curvature of the original hypocycloid) = 8Oq.



3. *Length of Curve.*—Referring to the same figure as for radius of curvature, an element of the length of curve is equal to the product of the radius of curvature into the angle between the normals. Now if μM be a diameter of the rolling circle or hoop, μt a tangent to the hypocycloid, and we take an arc tu = MM′, where M′ is a contiguous point to M on the apocentric circle; we shall have $\angle t\mathrm{M}u = 3$ times $\angle$ between the normals to the hypocycloid through M, M′; and Mt = radius of curvature; hence if μt meet Mu in τ, element of curve = $\frac{2}{3}$(Mt . $\angle t\mathrm{M}\tau$) = $\frac{2}{3} t\tau = \frac{2}{3}$ difference of chords μt, μu; therefore the length of the curve between cusp and apse = $\frac{4}{3}$ diameter of rolling circle = $\frac{8}{3}r$, where r is the radius of the pericentric circle; and the whole length of the curve is 6 times that between cusp and apse = 16r.



4. *Area.*—Let μt, $\mu' t'$ be two consecutive tangents; μ, μ' their centres; s, s' their vertices; and t, t' their points of contact; then we may suppose t, t' ultimately to coincide; sts', $\mu t\mu'$ to be triangles; $\mu s = \mu t$, and $\mu' s' = \mu' t'$; hence

$$\triangle \mu t \mu' = \tfrac{1}{4} sts' = \tfrac{1}{3} s\mu\mu' s'.$$

Proceeding from cusp to apse, the area between the pericentric circle and the hypocycloid, from the cuspidal tangent to the apse-point, is one-third of the semicircle; hence the area between the hypocycloid and the circle is equal to the area of the circle, and the whole area, including the circle, is double the area of the circle, or $2\pi r^2$.

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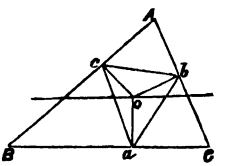
1704. (Proposed by R. FULLERTON.)—From any point in a straight line perpendiculars are drawn on the sides of a triangle, and the feet of these perpendiculars joined; find the locus of the centre of gravity (or *centroid*) of the triangle thus formed.

Solution by J. DALE; E. FITZGERALD; *and others.*



Let O be any point in the straight line $l\alpha+m\beta+n\gamma=0$; then drawing Oa, Ob, Oc perpendicular to BC, CA, AB, the coordinates (x, y, z) of the centre of gravity of the triangle abc will be given by the following equations,

$$2\alpha+\cos C\,.\,\beta+\cos B\,.\,\gamma=3x,$$
$$\cos C\,.\,\alpha+2\beta+\cos A\,.\,\gamma=3y,$$
$$\cos\beta\,.\,\alpha+\cos A\,.\,\beta+2\gamma=3z;$$

and we have given $l\alpha+m\beta+n\gamma=0$.

Eliminating α, β, γ, between these equations, we obtain

$$l\begin{vmatrix} x, & \cos C, & \cos B \\ y, & 2, & \cos A \\ z, & \cos A, & 2 \end{vmatrix} + m\begin{vmatrix} x, & \cos B, & 2 \\ y, & \cos A, & \cos C \\ z, & 2, & \cos B \end{vmatrix} + n\begin{vmatrix} x, & 2, & \cos C \\ y, & \cos C, & 2 \\ z, & \cos B, & \cos A \end{vmatrix} = 0.$$

This shows that the locus of the *centroid* is a straight line; its equation, in trilinear coordinates (x, y, z), may also be written as follows; viz.,

$$\left\{l(4-\cos^2 A)+m(\cos A\cos B-2\cos C)+n(\cos A\cos C-2\cos B)\right\}x$$
$$+\left\{l(\cos B\cos A-2\cos C)+m(4-\cos^2 B)+n(\cos B\cos C-2\cos A)\right\}y$$
$$+\left\{l(\cos C\cos A-2\cos B)+m(\cos C\cos B-2\cos A)+n(4-\cos^2 C)\right\}z=0.$$

[*Otherwise:* Taking the given line as axis of x, the coordinates of the feet of the perpendiculars drawn from a point $(h, 0)$ in it on the lines

$$a_1 \equiv x\cos\alpha_1+y\sin\alpha_1-p_1=0, \quad a_2=0, \quad a_3=0,$$

that is, on the sides of the given triangle, are

$$x_1=h+(p_1-h\cos\alpha_1)\cos\alpha_1, \quad y_1=(p_1-h\cos\alpha_1)\sin\alpha_1; \quad \&c.; \ \&c.;$$

hence the coordinates (x, y) of the centre of gravity of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) will be

$$x=\tfrac{1}{3}h\Sigma\sin^2\alpha+\tfrac{1}{3}\Sigma p\cos\alpha, \quad y=\tfrac{1}{3}\Sigma p\sin\alpha-\tfrac{1}{3}h\Sigma\sin\alpha\cos\alpha.$$

Eliminating h, we find that the locus of (x, y) is the straight line

$$(3x-\Sigma p\cos\alpha)\Sigma\sin\alpha\cos\alpha=(\Sigma p\sin\alpha-3y)\Sigma\sin^2\alpha.]$$

1715. (Proposed by J. GRIFFITHS, M.A.)—Let P be the point of intersection of the perpendiculars of any triangle ABC, O the centre of the circumscribed circle, M the middle point of PO, and O′ the centre of any one of the escribed circles of the triangles ABC, BPC, CPA, APB. Prove that the nine-point circle of the above triangle touches the common tangent of the circle (O′) and the circle described on MO′ as diameter.

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Solution by J. DALE; H. MURPHY; *and others.*

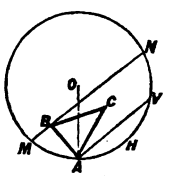
Here M is the centre of the nine-point circle of the triangle ABC, and it is known (see the solution of Mr. Wilkinson's Prize Question, in the *Lady's Diary* for 1855) that this circle touches each of the escribed circles above mentioned. Moreover the circle passing through the centres M, O′ of two circles which touch each other, and having its centre on the line MO′, touches the common tangents of these circles.

1333. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Three points being taken at random within a sphere as the corners of a plane triangle, determine the probability that it shall be acute.

Solution by M. W. CROFTON, B.A.

We shall first prove that one of the points may be fixed on the surface of the sphere without altering the probability in question.

In any position of the three points, one of them is farthest from the centre; call its distance from the centre ρ; then the other two points are confined within a concentric sphere of radius ρ. Let P be the probability of an acute triangle when one point is thus confined to the surface of a sphere, radius ρ, and the other two within that sphere. This probability P is clearly independent of the value of ρ. Now, taking the given sphere, we shall get all the possible cases by making the *farthest point* move over every point of its volume, giving, in each of its positions, every possible position to the two nearer points. And *in each of its positions* the probability of the two nearer ones forming with it an acute triangle is P. Therefore the total probability is equal to P.



Let one of the points A be fixed on the surface of the sphere; if two other points B, C are taken at random inside it, we shall find separately the probability of each angle A, B, C being obtuse; as the events are mutually exclusive, the probability of the triangle ABC being *obtuse* will be the sum of these three probabilities.

1. To find the chance that A is obtuse, let us fix B; then, drawing the plane AV perpendicular to AB, the chance required is $\frac{\text{volume of segment AHV}}{\text{volume of sphere}}$.

Let $r = \text{OA} = \text{radius of sphere}$, $\rho = \text{AB}$, $\theta = \angle \text{OAB}$; then segment AHV $= \frac{1}{3}\pi r^3 (1 - \cos\theta)^2 (2 + \cos\theta)$, therefore chance $= \frac{1}{4}(1-\cos\theta)^2(2+\cos\theta)$. Now, supposing B to move over the whole volume of the sphere, we have for the probability (P_A) that A is obtuse

$$\text{P}_\text{A} = \frac{1}{\text{sphere}} \iiint \tfrac{1}{4}(1-\cos\theta)^2(2+\cos\theta)\, dV$$

$$= \frac{3}{8r^3} \int_0^{\frac{1}{2}\pi} \int_0^{2r\cos\theta} (2 - 3\cos\theta + \cos^3\theta)\, \rho^2 \sin\theta\, d\theta\, d\rho.$$

The integration is easily performed, and we find $\text{P}_\text{A} = \frac{3}{70}$.

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2. To find the chance (P_B) that B is obtuse. Fix B as before; then the chance that B is acute is $\frac{\text{segment MHN}}{\text{sphere}}$.

Now, volume of MHN $= \frac{1}{3}\pi r^3 \left(\frac{\rho}{r}+1-\cos\theta\right)^2\left(2+\cos\theta-\frac{\rho}{r}\right)$,

$\therefore$ chance $= \frac{1}{4}\left\{2-3\cos\theta+\cos^3\theta+3\frac{\rho}{r}(1-\cos^2\theta)+3\frac{\rho^2}{r^2}\cos\theta-\frac{\rho^3}{r^3}\right\}$.

Hence the total probability $(1-P_B)$ that B is acute is, as before,

$$\frac{3}{8r^3}\int_0^{\frac{1}{2}\pi}\int_0^{2r\cos\theta}\left\{2-3\cos\theta+\cos^3\theta+3\frac{\rho}{r}(1-\cos^2\theta)+3\frac{\rho^2}{r^2}\cos\theta-\frac{\rho^3}{r^3}\right\}\rho^2\sin\theta\,d\theta\,d\rho,$$

whence, by very easy integrations, we find $1-P_B=\frac{53}{70}$, or $P_B=\frac{17}{70}$.

The probability that C is obtuse is of course the same as for B; hence the probability that the triangle is obtuse is

$$P_A+P_B+P_C=\frac{3}{70}+\frac{17}{70}+\frac{17}{70}=\frac{37}{70};$$

and therefore the probability that it is acute is $\frac{33}{70}$.

[Mr. CROFTON's method may be readily applied to show that if three points are taken at random within a *circle*, the probability of their being the vertices of an acute triangle is $\frac{4}{\pi^2}-\frac{1}{8}$. Mr. WOOLHOUSE's solution for the *sphere* is given on p. 234 of the *Educational Times* for January, 1863, and for the *circle*, on p. 22, Vol. I. of the *Reprint*.]

1692. (Proposed by H. J. PURKISS, B.A.)—Prove that, n being any positive integer,

$$\frac{2^{2n}}{\lfloor 2n+1}=\frac{1}{\lfloor 2n+1}+\frac{1}{\lfloor 2n-1}\frac{1}{\lfloor 2}+\frac{1}{\lfloor 2n-3}\frac{1}{\lfloor 4}+\cdots\cdots\cdots+\frac{1}{\lfloor 2n},$$

$$\frac{2^{4n-2}}{\lfloor 4n}=\frac{1}{2}\frac{1}{(\lfloor 2n)^2}+\frac{1}{\lfloor 2n-2}\frac{1}{\lfloor 2n+2}+\frac{1}{\lfloor 2n-4}\frac{1}{\lfloor 2n+4}+\cdots+\frac{1}{\lfloor 4n},$$

$$=\frac{1}{\lfloor 2n-1}\frac{1}{\lfloor 2n+1}+\frac{1}{\lfloor 2n-3}\frac{1}{\lfloor 2n+3}+\cdots\cdots\cdots+\frac{1}{\lfloor 4n-1},$$

$$\frac{2^{4n}}{\lfloor 4n+2}=\frac{1}{2}\frac{1}{(\lfloor 2n+1)^2}+\frac{1}{\lfloor 2n-1}\frac{1}{\lfloor 2n+3}+\cdots\cdots\cdots\cdots+\frac{1}{\lfloor 4n+1},$$

$$=\frac{1}{\lfloor 2n}\frac{1}{\lfloor 2n+2}+\frac{1}{\lfloor 2n-2}\frac{1}{\lfloor 2n+4}+\frac{1}{\lfloor 2n-4}\frac{1}{\lfloor 2n+6}+\cdots+\frac{1}{\lfloor 4n+2}.$$

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Solution by the PROPOSER.

Substituting the series for $\sin 2\theta$, $\sin\theta$, and $\cos\theta$ in the formula $\sin 2\theta = 2\sin\theta\cos\theta$, and equating coefficients of θ^{2n+1}, we get the first result. In a similar way, starting with $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, and equating coefficients of θ^{4n} and θ^{4n+2}, we get the four other results.

1750. (Proposed by W. K. CLIFFORD.)—Given four straight lines whose equations are connected by the syzygy $x+y+z+w=0$; show that the straight lines

$$lx+my+nz+sw=0, \quad \lambda x+\mu y+\nu z+\sigma w=0,$$

will be conjugates in respect of any conic touching (x, y, z, w), if

$$(l\mu+\lambda m)+(n\sigma+\nu s)=(l\nu+\lambda n)+(s\mu+\sigma m)=(l\sigma+\lambda s)+(m\nu+\mu n).$$

Show also that if a quadrangle be formed from the quadrilateral by taking the pole of each line in respect of the triangle formed by the other three, then the relation between the two figures will be reciprocal; and if two straight lines be conjugates in respect of any conic inscribed in the quadrilateral, their poles in respect of the common connector-triangle will be conjugates in respect of any conic circumscribing the quadrangle.

Solution by W. S. BURNSIDE, B.A.

1. Let $L=0$, $M=0$, $N=0$ be the equations of the diagonals of the quadrilateral formed by the lines x, y, z, w; then we have $x = L-M-N$, $y=-L+M-N$, $z=-L-M+N$, giving $w=L+M+N$; also the equation of any conic inscribed in the quadrilateral is of the form $aL^2+bM^2+cN^2=0$, where $A+B+C=0$ (A, B, C, &c., being the coefficients of the reciprocal conic as usual; see Salmon's *Conics*, Arts. 278 and 287). Now, reducing $lx+my+nz+sw$ and $\lambda x+\mu y+\nu z+\sigma w$ to the forms $L\alpha+M\beta+N\gamma$ and $L\alpha'+M\beta'+N\gamma'$, we have $\alpha=l-m-n+s$, $\beta=-l+m-n+s$, $\gamma=-l-m+n+s$; and similarly, α', β', γ' are expressed in terms of $\lambda, \mu, \nu, \sigma$. The condition that these lines should be conjugates with respect to $aL^2+bM^2+cN^2=0$ is $A\alpha\alpha'+B\beta\beta'+C\gamma\gamma'=0$; hence by making $\alpha\alpha'=\beta\beta'=\gamma\gamma'$, we satisfy the last condition when $A+B+C=0$: but this is the relation given in the Question, as is seen by writing for α, β, γ and α', β', γ' their values; in fact, $\alpha\alpha'+\beta\beta'=(s-n)(\sigma-\nu)+(l-m)(\lambda-\mu)=(l\lambda+m\mu+n\nu+s\sigma)-(l\mu+m\lambda)-(n\sigma+\nu s)$, $\beta\beta'+\gamma\gamma'=$ &c., &c.

2. The coordinates (referred to L, M, N) of the four points forming the quadrangle derived from the quadrilateral, as indicated in the Question, are $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$, $(1,1,1)$; and the equations of the lines joining these points two and two are $L \pm M=0$, $M \pm N=0$, $N \pm L=0$. Now, considering the triangle formed from the first three points, the equations of its sides are $M+N=0$, $N+L=0$, $L+M=0$; and plainly the *polar* of the fourth point $(1, 1, 1)$ is given by the equation $L+M+N=0$ or $w=0$; and similarly for the other triangles; which proves the reciprocity of the quadrilateral and quadrangle.

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3. The polars of the points $(1, \pm 1, \pm 1)$, with regard to the conic $L^2+M^2+N^2=0$, are the four lines $L \pm M \pm N = 0$; hence we have another reciprocal relation between the quadrangle and the quadrilateral. Moreover the pole of any line with regard to the conic $L^2+M^2+N^2=0$, and the *common connector* triangle (L, M, N) is the same; this being so, the third statement in the Question may be written as follows:—If two lines be conjugates with respect to any conic inscribed in the *quadrilateral*, their poles (in the ordinary sense) will be conjugates with respect to any conic circumscribing the *quadrangle;* the truth of which is manifest.

4. By means of the relations $(l\mu+\lambda m)+(n\sigma+\nu s) = (l\nu+\lambda n)+(s\mu+\sigma m) = (l\sigma+\lambda s)+(m\nu+\mu n)$, combined with the identical relation between $\lambda, \mu, \nu, \sigma$ of the form $a\lambda+b\mu+c\nu+d\sigma=0$, we may express $\lambda, \mu, \nu, \sigma$ in terms of l, m, n, s, involving the latter in the second degree; and hence it might be shown that when one of the lines passes through a fixed point, the *conjugate* line touches a conic inscribed in the triangle formed by the diagonals of the quadrilateral x, y, z, w. This also appears from writing the equations of the two lines in the forms $L\alpha+M\beta+N\gamma=0$ and $\frac{L}{\alpha}+\frac{M}{\beta}+\frac{N}{\gamma}=0$.

1752. (Proposed by Professor SYLVESTER.)—Prove that any proper fraction may be expanded under the form $\frac{e_1}{2}+\frac{e_2}{2^2}+\frac{e_3}{2^3}+\ldots+\frac{e_4}{2^4}+\ldots.$

where the e's are all either zero or unity, and moreover become periodic after a certain point in the series; and show thereby that a calculating machine capable of multiplying and extracting square roots, will serve for the extraction of any root whatever.

Solution by E. L. ROSOLIS.

Let $\frac{a}{b}$ be any proper fraction, and let e_1 denote the integral part of $\frac{2a}{b}$, and $\frac{a_1}{b}$ the remainder; then $\frac{a}{b}=\frac{1}{2}\times\frac{2a}{b}=\frac{1}{2}\left(e_1+\frac{a}{b}\right)$.

Again let e_2 be the integral part of $\frac{2a_1}{b}$, and $\frac{a_2}{b}$ the remainder; then

$$\frac{a_1}{b}=\frac{1}{2}\left(e_2+\frac{a_2}{b}\right);\ \text{and, in like manner, } \frac{a_2}{b}=\frac{1}{2}\left(e_3+\frac{a_3}{b}\right),\ \&c.$$

therefore $$\frac{a}{b}=\frac{e_1}{2}+\frac{e_2}{2^2}+\frac{e_3}{2^3}+\ldots.+\frac{e_n}{e^n}+\ldots.,$$

where the e's are either zero or unity, since a is less than b, and they become periodic, when b is not a power of 2, just as ordinary fractions become circulating decimals in certain cases.

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Hence $m^{\frac{1}{n}} = m^{\frac{e_1}{2}+\frac{e_2}{2^2}+\frac{e_3}{2^3}+\ldots} = \sqrt{m^{e_1}\sqrt{m^{e_2}\sqrt{m^{e_3}\ldots\ldots}}}$

therefore the nth root of a number m can be extracted by multiplying and extracting square roots.

1760. (Proposed by J. GRIFFITHS, M.A.)—Let each angular point of any triangle ABC be joined with each of two given points; and let α, β, γ; α', β', γ' denote the six points of intersection of the joining lines with the opposite sides of the triangle. It is required to prove that the conic which touches the sides of the two triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ touches also the two axes of homology of the triangles ABC, $\alpha\beta\gamma$, and ABC, $\alpha'\beta'\gamma'$. Show also that the triangle ABC is self-conjugate with respect to this conic.

What is the reciprocal of this theorem?

Solution by W. S. BURNSIDE, B.A.

Taking ABC as triangle of reference, the equations of the sides of the triangle $\alpha\beta\gamma$ and its axis of homology are of the form $lx \pm my \pm nz = 0$, and the conic $ax^2 + by^2 + cz^2 = 0$ will touch these four lines if $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = 0$; similarly the same conic will touch the sides and axis of homology of the triangle $\alpha'\beta'\gamma'$ if $\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c} = 0$, the equations of these lines being $l_1x \pm m_1y \pm n_1z = 0$; hence the conic given by the equation $\frac{x^2}{m^2n_1^2 - m_1^2n^2} + \frac{y^2}{n^2l_1^2 - n_1^2l^2} + \frac{z^2}{l^2m_1^2 - l_1^2m^2} = 0$ touches the eight lines $lx \pm my \pm nz = 0$, $l_1x \pm m_1y \pm n_1z = 0$; and with regard to this conic the triangle ABC is self-conjugate.

The reciprocal of this theorem is as follows:—

If, of the five quadrilaterals formed by five lines, two be taken; representing their common triangle by (abc), and the triangles formed by their diagonals respectively by $(a_1b_1c_1)$, $(a_2b_2c_2)$; a conic circumscribes the triangles $(a_1b_1c_1)$, $(a_2b_2c_2)$, and also passes through the two points, where the lines aa_1, bb_1, cc_1, and aa_2, bb_2, cc_2 are concurrent; the triangle (abc) being self-conjugate with respect to the conic.

1775. (Proposed by W. K. CLIFFORD.)—If a straight line meet the faces of the tetrahedron ABCD in the points a, b, c, d, respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

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Solution by PROFESSOR CAYLEY; *and* H. M. TAYLOR, B.A.

1. Let the given line be taken for the axis of z; the axes of x, y being any rectangular axes in the plane perpendicular thereto; the equations of the given line are therefore $(x=0, y=0)$. Take (a_1, β_1, γ_1), (a_2, β_2, γ_2), (a_3, β_3, γ_3), (a_4, β_4, γ_4) for the coordinates of the points A, B, C, D respectively; and $(0, 0, c_1)$, $(0, 0, c_2)$, $(0, 0, c_3)$, $(0, 0, c_4)$ for the coordinates of the points a, b, c, d respectively. Then, to determine c_1, the equation of the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ a_2, & \beta_2, & \gamma_2, & 1 \\ a_3, & \beta_3, & \gamma_3, & 1 \\ a_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

and cutting this by the line $x=0, y=0$, we have

$$\begin{vmatrix} 0, & 0, & c_1, & 1 \\ a_2, & \beta_2, & \gamma_2, & 1 \\ a_3, & \beta_3, & \gamma_3, & 1 \\ a_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

with similar equations for c_2, c_3, c_4 respectively. The four equations may be united into the single equation

$$\begin{vmatrix} c_1p_1, & 1, & a_1, & \beta_1 \\ c_2p_2, & 1, & a_2, & \beta_2 \\ c_3p_3, & 1, & a_3, & \beta_3 \\ c_4p_4, & 1, & a_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} p_1, & a_1, & \beta_1, & \gamma_1 \\ p_2, & a_2, & \beta_2, & \gamma_2 \\ p_3, & a_3, & \beta_3, & \gamma_3 \\ p_4, & a_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

where p_1, p_2, p_3, p_4 are arbitrary multipliers. Hence, writing successively $(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ and $(p_1, p_2, p_3, p_4) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$; we have *first*

$$\begin{vmatrix} c_1, & 1, & a_1, & \beta_1 \\ c_2, & 1, & a_2, & \beta_2 \\ c_3, & 1, & a_3, & \beta_3 \\ c_4, & 1, & a_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} 1, & a_1, & \beta_1, & \gamma_1 \\ 1, & a_2, & \beta_2, & \gamma_2 \\ 1, & a_3, & \beta_3, & \gamma_3 \\ 1, & a_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is,

$$\begin{vmatrix} 1, & a_1, & \beta_1, & c_1+\gamma_1 \\ 1, & a_2, & \beta_2, & c_2+\gamma_2 \\ 1, & a_3, & \beta_3, & c_3+\gamma_3 \\ 1, & a_4, & \beta_4, & c_4+\gamma_4 \end{vmatrix} = 0;$$

and *secondly*,

$$\begin{vmatrix} c_1\gamma_1, & 1, & a_1, & \beta_1 \\ c_2\gamma_2, & 1, & a_2, & \beta_2 \\ c_3\gamma_3, & 1, & a_3, & \beta_3 \\ c_4\gamma_4, & 1, & a_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} \gamma_1, & a_1, & \beta_1, & \gamma_1 \\ \gamma_2, & a_2, & \beta_2, & \gamma_2 \\ \gamma_3, & a_3, & \beta_3, & \gamma_3 \\ \gamma_4, & a_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is,

$$\begin{vmatrix} 1, & a_1, & \beta_1, & c_1\gamma_1 \\ 1, & a_2, & \beta_2, & c_2\gamma_2 \\ 1, & a_3, & \beta_3, & c_3\gamma_3 \\ 1, & a_4, & \beta_4, & c_4\gamma_4 \end{vmatrix} = 0;$$

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and these two results may be united into the single formula

$$\begin{Vmatrix} 1, & a_1, & \beta_1, & c_1+\gamma_1, & c_1\gamma_1 \\ 1, & a_2, & \beta_2, & c_2+\gamma_2, & c_2\gamma_2 \\ 1, & a_3, & \beta_3, & c_3+\gamma_3, & c_3\gamma_3 \\ 1, & a_4, & \beta_4, & c_4+\gamma_4, & c_4\gamma_4 \end{Vmatrix} = 0.$$

Now the equation of a sphere having for the extremities of a diameter the points (a, β, γ) and (a, b, c) is

$$[x-\tfrac{1}{2}(a+a)]^2+[y-\tfrac{1}{2}(b+\beta)]^2+[z-\tfrac{1}{2}(c+\gamma)]^2 = \tfrac{1}{4}[(a-a)^2+(b-\beta)^2+(c-\gamma)^2],$$

or, $$(x-a)(x-a)+(y-b)(y-\beta)+(z-c)(z-\gamma)=0,$$

or, $$x^2+y^2+z^2-(a+a)x-(b+\beta)y-(c+\gamma)z+aa+b\beta+c\gamma=0;$$

therefore, when the two points are (a, β, γ) and $(0, 0, c)$, the equation is

$$x^2+y^2+z^2-ax-\beta y-(c+\gamma)z+c\gamma=0.$$

Hence, putting for shortness $P = -ax-\beta y-(c+\gamma)z+c\gamma$, viz., $P_1 = -a_1x-\beta_1y-(c_1+\gamma_1)z+c_1\gamma_1$, &c., the equations of the four spheres are

$$x^2+y^2+z^2+P_1=0, \quad x^2+y^2+z^2+P_2=0, \quad x^2+y^2+z^2+P_3=0,$$
$$x^2+y^2+z^2+P_4=0,$$

and the four spheres will have a common radical axis, if for proper values of the multipliers μ, ν, ρ we have

$$\mu(P_1-P_2)+\nu(P_1-P_3)+\rho(P_1-P_4)=0,$$

or what is the same thing, if for proper values of λ, μ, ν, ρ we have

$$\lambda P_1+\mu P_2+\nu P_3+\rho P_4=0, \quad \lambda+\mu+\nu+\rho=0;$$ that is, if

$$\lambda+\mu+\nu+\rho=0, \quad \lambda a_1+\mu a_2+\nu a_3+\rho a_4=0, \quad \lambda\beta_1+\mu\beta_2+\nu\beta_3+\rho\beta_4=0,$$
$$\lambda(c_1+\gamma_1)+\mu(c_2+\gamma_2)+\nu(c_3+\gamma_3)+\rho(c_4+\gamma_4)=0,$$
$$\lambda c_1\gamma_1+\mu c_2\gamma_2+\nu c_3\gamma_3+\rho c_4\gamma_4=0;$$

and eliminating from these equations $(\lambda, \mu, \nu, \rho)$, we find the above mentioned relation between $a_1, \beta_1, \gamma_1, c_1$, &c.; which proves the theorem.

2. The corresponding theorem *in plano* is, that the three circles having for their respective diameters the diagonals of any quadrilateral, have a common radical axis. This may be proved by a similar analysis, or somewhat differently, in part geometrically, as follows:—Consider the circle having for the extremities of a diameter the points (a, β), (α, b) respectively; the equation of the circle is $(x-a)(x-\alpha)+(y-b)(y-\beta)=0$. Hence, taking the three circles which correspond to the points (a_1, β_1), (α_1, b_1); (a_2, β_2), (α_2, b_2); (a_3, β_3), (α_3, b_3), these will have a common radical axis, if only

$$\lambda+\mu+\nu=0, \quad \lambda(a_1+\alpha_1)+\mu(a_2+\alpha_2)+\nu(a_3+\alpha_3)=0,$$
$$\lambda(b_1+\beta_1)+\mu(b_2+\beta_2)+\nu(b_3+\beta_3)=0,$$
$$\lambda(a_1\alpha_1+b_1\beta_1)+\mu(a_2\alpha_2+b_2\beta_2)+\nu(a_3\alpha_3+b_3\beta_3)=0;$$

or, what is the same thing, introducing a new arbitrary quantity σ, if only

$$\lambda+\mu+\nu=0, \quad \lambda(a_1+\alpha_1)+\mu(a_2+\alpha_2)+\nu(a_3+\alpha_3)=0,$$
$$\lambda(b_1+\beta_1)+\mu(b_2+\beta_2)+\nu(b_3+\beta_3)=0,$$
$$\lambda a_1\alpha_1+\mu a_2\alpha_2+\nu a_3\alpha_3=\sigma, \quad \lambda b_1\beta_1+\mu b_2\beta_2+\nu b_3\beta_3=-\sigma.$$

Hence we have *first* $$\begin{vmatrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \end{vmatrix} = 0;$$

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and then, this equation being satisfied, we have simultaneously

$$(\lambda, \mu, \nu) = \theta \begin{Vmatrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \end{Vmatrix} = \phi \begin{Vmatrix} 1, & 1, & 1 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \end{Vmatrix},$$

where of course the value of $(\theta : \phi)$ can be found by comparing the two values of λ, μ, or ν; thus, comparing the values of λ, we have

$$\theta(a_3+\alpha_3-a_2-\alpha_2) = \phi(b_3+\beta_3-b_2-\beta_2).$$

Substituting the first set of values in the fourth equation, and the second set in the fifth equation, and adding, we find *secondly*

$$\theta \begin{vmatrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ a_1\alpha_1, & a_2\alpha_2, & a_3\alpha_3 \end{vmatrix} + \phi \begin{vmatrix} 1, & 1, & 1 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \\ b_1\beta_1, & b_2\beta_2, & b_3\beta_3 \end{vmatrix} = 0,$$

where $\theta : \phi$ is a given ratio.

Now if (a_1, β_1), (a_1, b_1), (a_2, β_2), (a_2, b_2), (a_3, β_3), (a_3, b_3) are the six angles of a quadrilateral; then, *first*, by a known theorem, the middle points of the three diagonals are in a line; that is, we have

$$\begin{vmatrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \end{vmatrix} = 0;$$

and, *secondly*, by the theorem that any pencil of lines drawn to the angles of a quadrilateral are a pencil in involution, applying this to the pencils of lines parallel to the two axes respectively, we have

$$\begin{vmatrix} 1, & 1, & 1 \\ a_1+\alpha_1, & a_2+\alpha_2, & a_3+\alpha_3 \\ a_1\alpha_1, & a_2\alpha_2, & a_3\alpha_3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1, & 1, & 1 \\ b_1+\beta_1, & b_2+\beta_2, & b_3+\beta_3 \\ b_1\beta_1, & b_2\beta_2, & b_3\beta_3 \end{vmatrix} = 0.$$

The above mentioned first and second conditions are thus satisfied when the six points are the angles of a quadrilateral.

1780. (Proposed by A. RENSHAW.)—Find the locus of a point whose polar, in reference to a given conic, cuts off a given area from the axes.

Solution by W. H. LAVERTY; E. MCCORMICK; R. TUCKER, M.A.; E. FITZGERALD; *the* PROPOSER; *and others.*

Let $Lx + My + R = 0$, and $Lx' + My' + R = 0$, be the equations to a conic and to the polar of the point (x', y') with respect to that conic, where $(L=0)$, $(M=0)$ are the equations to the diameters bisecting chords parallel to the axes of x and y respectively, and $(R=0)$ is the polar of the origin. We may put the equation of the polar of (x', y') into the form $L'x + M'y + R' = 0$, and we have $-\frac{R'}{L'}$, $-\frac{R'}{M'}$, as the intercepts on the axes; therefore $\frac{R'^2}{L'M'} = \mu$ (= a constant), or, omitting the accents, the equation of the required locus is $R^2 = \mu . LM$, which is that of a conic touching the diameters $(L=0)$, and $(M=0)$ of the original conic, at the points where $(R=0)$ meets those diameters.

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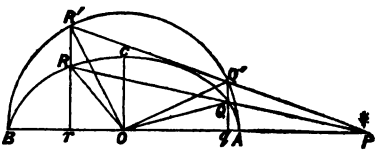
1781. (Proposed by E. CONOLLY.)—From a point P in the produced major axis of an ellipse to draw a line PQR, cutting the curve in Q and R, so that if Q and R be joined with the centre O of the ellipse, the triangle OQR may be a maximum.

Solution by M. COLLINS, B.A.; W. H. LAVERTY; H. M. TAYLOR, B.A.; E. FITZGERALD; *and many others.*

Draw Qq, Rr perpendicular to the major axis AB, meeting the semicircle on AB in Q', R'; then, OC being the minor semi-axis, $R'r : Rr = OA : OC = Q'q : Qq$; hence $R'Q'$ passes through P;



$\therefore$ $OA : OC = \triangle POR' : POR = \triangle POQ' : POQ = \triangle OQ'R : OQR$;

hence the triangle OQR is greatest when $OQ'R'$ is greatest, which will plainly be the case when $Q'OR'$ is a right angle, or when $OQ'R'$ and $OR'Q'$ are each half a right angle; therefore the circle drawn round the square whose side is OP will cut the semi-circle $AQ'R'B$ in the point Q' (or R'), from which drawing $Q'q$ (or $R'r$) perpendicular to AB, it will meet the given ellipse in Q (or R) which determines the required position of PQR.

1786. (Proposed by Dr. BOOTH, F.R.S.)—If $4a$ be the parameter of a parabola, S_y the length (measured from the vertex) of an arc the ordinate of whose extremity is y, and R_y the radius of curvature at the extremity, prove that $S_{4a} - 3S_a = R_a$.

Solution by R. WARREN, B.A.; E. FITZGERALD; M. COLLINS, B.A.; E. MCCORMICK; *and many others.*

By well-known formulæ, we have

$$R_y = \frac{(y^2+4a^2)^{\frac{3}{2}}}{4a^2}, \quad S_y = \frac{y\,(y^2+4a^2)^{\frac{1}{2}}}{4a} + a \log \frac{y+(y^2+4a^2)^{\frac{1}{2}}}{2a};$$

therefore $R_a = \frac{5}{4}a\sqrt{5}$, $S_a = \frac{1}{4}a\sqrt{5} + a \log\left\{\frac{1}{2}(1+\sqrt{5})\right\}$,

$$S_{4a} = 2a\sqrt{5} + a \log(2+\sqrt{5}) = 2a\sqrt{5} + 3a \log\left\{\tfrac{1}{2}(1+\sqrt{5})\right\};$$

therefore $$S_{4a} - 3S_a = \tfrac{5}{4}a\sqrt{5} = R_a.$$

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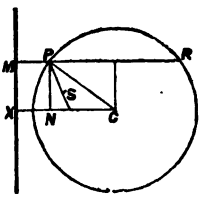
1771. (Proposed by Professor CAYLEY.)—Given a circle and a line, it is required to find a parabola, having the line for its directrix, and the circle for a circle of curvature.

Solution by H. M. TAYLOR, B.A.; F. D. THOMSON, M.A.; M. COLLINS, B.A.; R. TUCKER, M.A.; *and others.*

From the centre C of the given circle draw CX perpendicular to the given directrix. Trisect CX, and let N be one of the points of trisection; so that $CN = 2NX$. Draw NP at right angles to CX, meeting the circle in P; through P draw MPR parallel to CX, meeting the circle again in R; also draw PS making an angle SPC = RPC, and take $PS = \frac{1}{4}PR$. Then a parabola, with focus S and given directrix, will plainly touch the circle at P; and will have it as a circle of curvature, since in a parabola the chord of curvature parallel to the axis is equal to 4SP.



II. *Solution by the* PROPOSER.

Let $x^2+y^2-1=0$ be the equation of the given circle, $x=m$ that of the given line. Taking on the circle an arbitrary point $(\cos\theta, \sin\theta)$, we may find a parabola having the given line for its directrix, and touching the circle at the last mentioned point; viz., the equation of the parabola is found to be

$$y^2-2y\sin\theta\,(1+2\cos^2\theta-2m\cos\theta)-4x\cos^2\theta\,(\cos\theta-m) +1+3\cos^2\theta-4m\cos\theta=0.$$

[There is no difficulty in *verifying* that this parabola has for its directrix the line $x-m=0$, that the equation is satisfied by the values $x=\cos\theta$, $y=\sin\theta$, and that the derived equation is satisfied by the values

$$x=\cos\theta,\quad y=\sin\theta,\quad \frac{dy}{dx}=-\cot\theta].$$

Representing for a moment the left-hand side of the equation by U, we have identically

$$\begin{aligned}U-\cos^2\theta\,(x^2+y^2-1)&=y^2\sin^2\theta-x^2\cos^2\theta-2y\sin\theta\,(1+2\cos^2\theta-2m\cos\theta)-4x\cos^2\theta(\cos\theta-m)\\&\qquad+1+4\cos^2\theta-4m\cos\theta\\&=(y\sin\theta+x\cos\theta-1)\,(y\sin\theta-x\cos\theta-1-4\cos^2\theta+4m\cos\theta).\end{aligned}$$

Hence to find the intersections of the parabola with the circle, we have first

$$x^2+y^2-1=0,\quad y\sin\theta+x\cos\theta-1=0,$$

giving the point $(\cos\theta, \sin\theta)$ *twice*, since $y\sin\theta+x\cos\theta-1=0$ is the equation of the tangent to the circle at the point in question; and secondly

$$x^2+y^2-1=0,\quad y\sin\theta-x\cos\theta-1-4\cos^2\theta+4m\cos\theta=0,$$

giving the remaining two points of intersection. If the circle be a circle of curvature, one of these must coincide with the point $(\cos\theta, \sin\theta)$, that is the equation $y\sin\theta-x\cos\theta-1-4\cos^2\theta+4m\cos\theta=0$, must be satisfied by the values $x=\cos\theta$, $y=\sin\theta$; this will be the case if

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$-6\cos^2\theta + 4m\cos\theta = 0$, that is $\cos\theta = 0$, giving for the parabola $y^2 \pm 2y + 1 = 0$, which is not a proper Solution, or else $\cos\theta = \frac{2}{3}$, giving $\sin\theta = \pm\left(1-\frac{4}{9}m^2\right)^{\frac{1}{2}}$, so that there are *two* parabolas satisfying the conditions of the problem; if to fix the ideas we take the upper sign, the equation of the corresponding parabola is

$$y^2 - 2\left(1-\frac{4}{9}m^2\right)^{\frac{3}{2}}y + \frac{16}{27}m^3x + 1 - \frac{4}{3}m^2 = 0;$$

and it may be added that the coordinates of the focus are

$$x = m - \frac{8}{27}m^3, \quad y = \left(1-\frac{4}{9}m^2\right)^{\frac{3}{2}}.$$

The equation of the other parabola and the coordinates of the focus are of course found by merely changing the sign of the radical. The parabolas are real if $m < \frac{3}{2}$; if $m = \frac{3}{2}$ we have a single parabola, the point of contact being in this case the vertex of the parabola; and if $m > \frac{3}{2}$ the parabolas are imaginary.

[Professor Cayley states that he was led to the foregoing problem by the consideration of the curve (proposed for investigation in Quest. 1812) which is the envelope of a variable circle having its centre in the given circle and touching the given line. The required curve (which is of the sixth order) has two cusps which, it is easy to see geometrically, are the foci of the parabolas in the problem. Taking $(\cos\theta, \sin\theta)$ for the coordinates of the centre of the variable circle, we shall have

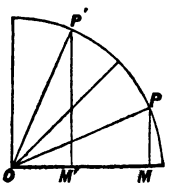
$$x = \tfrac{3}{2}\cos\theta - m\cos 2\theta + \tfrac{1}{2}\cos 3\theta, \; y = \tfrac{3}{2}\sin\theta - m\sin 2\theta + \tfrac{1}{2}\sin 3\theta,$$

for the coordinates of a point on the envelope.]

1773. (Proposed by M. W. Crofton, B.A.)—A traveller starts from a point on a straight river, and travels in a random direction a certain distance in a straight line. Having quite lost his way, he starts again at random the next morning, and travels the same distance; what is the chance of his reaching the river again in the second day's journey?

Solution (1) *by* H. M. Taylor, B.A.; (2) *by* E. McCormick; H. Murphy; M. Collins, B.A.; E. Fitzgerald; *the* Proposer; *and many others.*

1. Let OM be the side of the river, O the point from which the man starts, P his position at any point in the circle whose radius is the length of a day's journey. Then the chance of his reaching the river, supposing him to be at P, is $\frac{\angle \mathrm{OPM}}{2 \text{ rt. angles}}$; and, supposing him to be at P′, the chance of reaching the river is $\frac{\angle \mathrm{OP'M'}}{2 \text{ rt. angles}}$. Now let OPM + OP′M′ = one rt. angle; then the chance,



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on the supposition that he is at either P or P′, is $\frac{1 \text{ rt. angle}}{4 \text{ rt. angles}}$, or $\frac{1}{4}$.

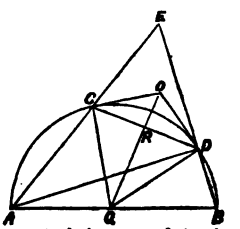
This is the chance required; since the number of ways we may choose pairs of positions P and P′ cannot affect the result, as all these pairs are equally likely to occur; P, P′ being equidistant from a line through O making half a right angle with OM.

2. *Otherwise:* Let OP (in the figure to the first Solution) be the first day's journey, and PM a perpendicular to the river's bank OM; then the probability that the inclination of OP to OM lies between θ and $\theta + \Delta\theta$ is clearly $\frac{\Delta\theta}{\pi}$; and the probability of reaching the river, from P, on the second day, is $\frac{2 \angle \text{OPM}}{4 \text{ rt. angles}}$ or $\frac{\pi - 2\theta}{2\pi}$; hence the whole probability required is

$$\Sigma\left(\frac{\pi-2\theta}{2\pi} \cdot \frac{\Delta\theta}{\pi}\right), \text{ or } 2\int_0^{\frac{1}{2}\pi} \frac{\pi-2\theta}{2\pi}\frac{d\theta}{\pi} = \frac{1}{\pi^2}\left(\pi\theta - \theta^2\right)_{\theta=0}^{\theta=\frac{1}{2}\pi} = \frac{1}{4}.$$

1774. (Proposed by H. MURPHY.)—If the ends of a fixed diameter of a circle be joined to the ends of a chord variable in position, but of given magnitude, the locus of the centre of the circle circumscribing the triangle formed by the chord and the segments of the joining lines outside the circle, is a circle concentric with the given one.

Solution by G. O. HANLON; E. CONOLLY; W. H. LAVERTY; E. MCCORMICK; R. TUCKER, M.A.; E. FITZGERALD; *the* PROPOSER; *and many others.*



Let Q be the centre of the given circle, AB the fixed diameter, CD the chord given in magnitude, E the intersection of AC, BD, and O the centre of the circle drawn round the triangle CDE. Then since the angles COR, CED are equal, their complements OCD, CAD are also equal; hence OC, OD are tangents to the given circle; therefore the quadrilateral OCQD, right-angled at C and D, is invariable in form, and the theorem is obvious.

[If AB be *any fixed chord*, the angle E is constant, being equal to the difference between the constant angles BDA and DAC; hence the locus of E is a circular segment passing through A and B.]

1060. (Proposed by EXHUMATUS.)—Two coins are thrown at random on a round table. What is the chance that they will lie upon a *diameter* of the table?

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Solution by the EDITOR.

Let O be the centre of the table, r its radius; a, b the radii of the coins; $2\alpha, 2\beta$ the angles subtended at O by the coins (a), (b), when they lie on the table at their greatest possible distance from O; also let 2θ be the angle subtended at O by the coin (a) when its centre is at a distance x from O.

We shall regard the coins as constrained to lie *wholly* on the table (as would be the case, for instance, if there were a rim round it); the calculation would of course be the same if we supposed the coins free to lie partly over the edge of the table.

Put $r-a=h$, $r-b=k$; then h, k are the radii of the circles, around O as centre, within which the centres of the coins (a), (b) must lie; and thus the probability that the centre of (a) will lie between the distances x and $x+\Delta x$, from O, is the ratio of the circular ring whose bounding radii are $(x, x+\Delta x)$ to the whole circle (h), that is $2\pi x\Delta x : \pi h^2$, or $2x\Delta x : h^2$. Through O draw two tangents (QOQ', ROR', which include an angle $QOR = 2\theta$) to the circle (a); within the angle QOR' draw two lines (oq, or') parallel to OQ, OR', and at a distance b from them; draw also the like lines $(o'q', o'r)$ within the angle Q'OR; and let these parallels meet the circle (k) in q, r', q', r. Then it is clear that some part of both coins will lie on the same diameter of the table when (and *only* when) the centre of (b) falls anywhere on the area $qor'q'or$ $(= S$ suppose).

Now S is made up of four sectors whose angle is $(\theta+\beta)$, and four triangles whose altitude is b and base $b(\cot\beta-\tan\theta)$, hence we have

$$S = 2k^2(\theta+\beta)+2b^2(\cot\beta-\tan\theta).$$

Moreover when $\theta > \frac{1}{2}\pi-\beta$, or $x < a\sec\beta$, both coins *must* lie on a diameter of the table. Hence the whole probability (p) required is

$$p = \int_0^{a\sec\beta} \frac{2xdx}{h^2} + \int_{a\sec\beta}^{h} \frac{S}{\pi h^2}\cdot\frac{2xdx}{h^2} = \frac{a^2}{h^2}\sec^2\beta + \frac{2a^2}{\pi h^2}\int(\theta+\beta)\,.\,d(\operatorname{cosec}^2\theta)$$

$$+\frac{2a^2b^2}{\pi h^2k^2}\int\left\{\cot\beta\,.\,d(\operatorname{cosec}^2\theta)+2\operatorname{cosec}^2\theta d\theta\right\}$$

$$= \frac{a^2}{h^2}\sec^2\beta+\frac{2a^2}{\pi h^2}\left\{(\theta+\beta)\operatorname{cosec}^2\theta+\cot\theta\right\}+\frac{2a^2b^2}{\pi h^2k^2}\left\{\cot\beta\operatorname{cosec}^2\theta-2\cot\theta\right\}.$$

Taking these integrals between limits $(\theta = \frac{1}{2}\pi-\beta,\ \theta=\alpha)$, and putting for h, k their values $a\operatorname{cosec}\alpha$, $b\operatorname{cosec}\beta$, we obtain finally

$$p = \frac{1}{\pi}\left\{(2\alpha+2\beta)+\sin(2\alpha+2\beta)\right\}.$$

As a *numerical* example, let it be required to find the probability that two florins (diameters $1\frac{1}{4}$ inches, suppose,) thrown on a table 50 inches in diameter, will lie on a diameter of the table.

$$\text{Here}\quad \alpha=\beta=\sin^{-1}\left(\tfrac{1}{25}\right) = 1^\circ\,28'\,10'';$$

$$\therefore\ p = \frac{1}{\pi}(4\alpha+\sin 4\alpha) = \cdot06524 = \tfrac{1}{15}\text{ nearly, or }\tfrac{3}{46}\text{ more nearly, \&c.}$$

Hence the odds are about 14 to 1 *against* their resting on a diameter of the table.

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1059. (Proposed by EXHUMATUS.)—From the arms of a bent lever two pieces are cut off at random; what is the chance that, in the position of equilibrium, one arm will rise above the horizontal?

Solution by the EDITOR.

Let a, ka be the original lengths of the arms (k being not <1); x, y their lengths, respectively, after the cutting; and α the (*obtuse*) angle contained between them. Supposing the shortened arms, (x) and (y), to be moveable about a fulcrum at their point of intersection, we find by taking moments of their weights about this point, that (x) will be horizontal if $x \times \frac{1}{2}x = y \times \frac{1}{2}y \cos(\pi - \alpha)$, or, putting $\cos(\pi - \alpha) = \lambda^2$, if $x = \lambda y$; similarly (y) will be horizontal if $y = \lambda x$. Hence (y) will rise above the horizontal, for *all* values of x, if $y < \lambda x$; also (x) will rise above the horizontal if $x < \lambda y$ or $y > \frac{x}{\lambda}$, and this may be the case for *all* values of x if $\lambda k > 1$, but only for values of x up to $\lambda k a$ if $\lambda k < 1$. Moreover the probability that (a) will be cut between the distances (x, $x + \Delta x$) from the fulcrum being $\frac{\Delta x}{a}$, the whole probability (p_{ka}) that (y) will rise above the horizontal, is

$$\Sigma\left(\frac{\lambda x}{ka} \cdot \frac{\Delta x}{a}\right), \text{ or } p_{ka} = \frac{1}{ka^2}\int_0^a \lambda x\,dx = \frac{\lambda}{2k} \quad \ldots\ldots\ldots\ldots (1).$$

Similarly, the whole probability (p_a) that (x) will rise above the horizontal is

$$\text{when } ka > \frac{1}{\lambda}a, \quad p_a = \frac{1}{ka^2}\int_0^a \left(ka - \frac{x}{\lambda}\right) dx = 1 - \frac{1}{2\lambda k} \ldots\ldots\ldots\ldots (2),$$

$$\text{when } ka < \frac{1}{\lambda}a, \quad p_a = \frac{1}{ka^2}\int_0^{\lambda ka} \left(ka - \frac{x}{\lambda}\right) dx = \tfrac{1}{2}\lambda k \quad \ldots\ldots\ldots\ldots (3).$$

The probability that one or the other of the two arms will rest above the horizontal *after* the cutting is $\{(1) + (2)\}$ or $\{(1) + (3)\}$ according as the shorter arm (a) does or does not rest above the horizontal *before* the cutting.

If $\lambda k = 1$, that is, if the arm (a) rests *horizontally* before the cutting, it is an *even* chance that this arm will rest above the horizontal after the cutting; and the probability that one of the two arms will rise above the horizontal after the cutting is, in this case, $\frac{1}{2}(1 + \lambda^2)$ or $\sin^2 \frac{1}{2}\alpha$.

NOTES ON QUESTIONS 1623, 1649, 1679; BY F. D. THOMSON, M.A.

1. The following corollaries to the Solution of Question 1623 (*Reprint*, Vol. III., p. 39) contain *analytical* Solutions of the *general* cases of Quests. 1649, 1679. (Vol. III., pp. 58, 81). In Quest. 1623, it is proved that if ABC is a triangle inscribed in a conic; if chords Aa, Bb, Cc are drawn through the point O, the polar of which is PQR; and if Ab, Bc, Ca meet PQR in

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R, P, Q respectively: then, S being any point on the conic, SP, SQ, SR will meet the sides BC, CA, AB in three points in a straight line.

2. With the notation of the Solution it is shown that, $LM = R^2$ being the equation to the conic, and a, b, c, μ the quantities which determine A, B, C, S, the equation to the line required is

$$2\mu abcL + \left\{\mu^3 + \mu^2(a+b+c) - \mu(ab+bc+ca) - abc\right\} R - 2\mu^2 M = 0 \ldots\ldots \text{(i)}.$$

Now if SO meet the conic again in S′, the new line corresponding to (i) will be obtained by writing $-\mu$ for μ; thus, $\alpha'\beta'\gamma'$ being the new, $\alpha\beta\gamma$ the former line, the equation to $\alpha'\beta'\gamma'$ is

$$2\mu abcL + \left\{\mu^3 - \mu^2(a+b+c) - \mu(ab+bc+ca) + abc\right\} R + 2\mu^2 M = 0 \ldots\ldots \text{(ii)}.$$

The two lines $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ meet in a point whose locus is

$$abcR^2 = \left\{(ab+bc+ca)\,R - 2abcL\right\}\left\{(a+b+c)R - 2M\right\} \ldots\ldots\ldots \text{(iii)},$$

a conic to which $R = 0$ is the chord of contact of tangents through the point

$$R = \frac{2abcL}{ab+bc+ca} = \frac{2M}{a+b+c} \ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

Hence, when the original conic is projected into a circle, and R becomes the line at infinity, the conic (iii) will also become a circle of which the point (iv) is the centre.

3. If $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ meet PQR in X and Y, the equations to OX, OY will be

$$abcL - \mu M = 0, \quad abcL + \mu M = 0;$$

therefore OX, OY form an harmonic pencil with L and M: hence in the projection OX and OY become *two lines at right angles*, and the lines $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, which in the projection are parallel to OX and OY, become also two lines at right angles. This proves the result of Quest. 1649 so far as regards the perpendicularity.

4. It will be found that AP meets BC in the point given by

$$a^2R = (ab+bc+ca)\,R - 2abcL, \quad \frac{bc}{a}R = (a+b+c)R - 2M,$$

which evidently lies on the conic (iii). Hence, in the projection, the circle locus passes through the feet of the perpendiculars, and is therefore the nine-point circle.

5. It is easily seen that AP, BQ, CR meet in a point given by

$$R = \frac{abcL}{ab+bc+ca} = \frac{M}{a+b+c},$$

which is evidently in the same straight line with O and the point (iv). Hence, in the projection, the centre of the nine-point circle is in the same straight line with the centre of the given circle and the intersection of the perpendiculars.

6. It will be seen that the same locus (iii) will be obtained if we write $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ for a, b, c respectively, that is, if we take a triangle such that the equation to AB is $\frac{1}{ab}L - \left(\frac{1}{a} + \frac{1}{b}\right)R + M = 0$, but I do not see any simple geometrical interpretation of this.

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7. In order to find the envelope of the line $\alpha\beta\gamma$, given by equation (i), we have, if u, v, w be the coefficients of L, R, M respectively,

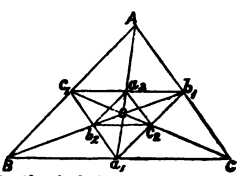
$$\frac{u}{2\mu abc} = \frac{v}{\mu^3+\mu^2(a+b+c)-\mu(ab+bc+ca)-abc} = \frac{w}{-2\mu^3};$$

$$\therefore\ w^3abc - w^2u(a+b+c) - wu^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \frac{u^3}{abc} = 2uvw \ldots\ldots \text{(vi)},$$

a *tangential* equation to the envelope, which is therefore a curve of the third class touching the line $w=0, u=0$, that is, the line R. Now the coordinates of the line BC are proportional to bc, $-(b+c)$, 1, and it will be found that they satisfy equation (vi). Hence, the envelope touches the sides of the given triangle. Again, the coordinates of CR will be seen to be proportional to abc, c^2-ab, $-c$, which satisfy (vi); therefore, *in the projection*, the envelope (vi) touches the line at infinity, the three sides of the triangle and the three perpendiculars. See the Solutions of Questions 1679, 1680. (*Reprint*, Vol. III., pp. 81—84.)

1727. (Proposed by E. CONOLLY.)—If the middle points of the sides of any triangle be joined, and the middle points of the sides of this new triangle also joined, and so on *ad infinitum*; the limit to which these triangles tend is a point, and the sum of the squares on the lines drawn therefrom to the angles of all the inscribed triangles is one-third of the sum of the squares on the lines drawn from the same point to the angles of the original triangle.

Solution by J. DALE; W. H. LAVERTY; *the* PROPOSER; *and others.*



It is evident from the construction that the lines Aa_1, Bb_1, Cc_1, which bisect BC, CA, AB respectively, also bisect the sides of all the inscribed triangles, which are respectively parallel to BC, CA, AB; and hence it follows that the ultimate limit of the inscribed triangles is an infinitesimal triangle, having its sides proportional to BC, CA, AB and the point O for its *centroid* (or centre of gravity). At the limit, the infinitesimal triangle and the point O may be supposed, without sensible error, to coincide. Now, by a well-known theorem, we have $Oa_1 = \frac{1}{2}OA$, $Oa_2 = \frac{1}{2}Oa_1 = \frac{1}{4}OA$, $Oa_3 = \frac{1}{8}OA$, and so on *ad infinitum*;

therefore $\Sigma Oa_1^2 = OA^2(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots) = \frac{1}{3}OA^2$;

similarly $\Sigma Ob_1^2 = \frac{1}{3}OB^2$, and $\Sigma Oc_1^2 = \frac{1}{3}OC^2$;

therefore $\Sigma Oa_1^2 + \Sigma Ob_1^2 + \Sigma Oc_1^2 = \frac{1}{3}(OA^2+OB^2+OC^2)$.

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1829. (Proposed by M. W. CROFTON, B.A.)—Two points being taken at random within (1) a circle or (2) a sphere, what is the probability that their distance apart is less than a given line?

Solution by PROFESSOR SYLVESTER.

1. *For the circle.* Call the radius 1, the length of the given line $2k$, the measure of the favourable cases F, and the required probability p.

(α). Block out from the given circle the interior concentric circle πk^2.

(β). Resolve the combinations which supply the remaining favourable cases into concentric rings, each of thickness dr, enclosing circles of radii varying from k to r, which must be *doubled* in order that each ring may be brought into combination with each that precedes as well as with each that follows it. Each such ring may be regarded as a circumference loaded with a uniform density $2\pi r dr$.

(γ). From any one of the circles thus enclosed block out a smaller circle πk^2, and resolve the remainder into rings of variable thickness, by drawing circles of all radii from k to r touching one another at the same point.

(δ). Divide each such ring into quadrilateral elements by drawing normals to the bounding contours all round.

The first circle blocked out supplies to F a portion $\pi^2 k^4$. The sets of circles subsequently blocked out supply a portion represented by

$$\int_k^1 4\pi r dr\,(\pi k^2) = 2\pi^2 k^2 - 2\pi^2 k^4.$$

The combinations remaining supply to M a portion represented by

$$\int_k^1 4\pi r dr \int_k^r \frac{dr}{r} \Sigma\,(pds),$$

where s denotes the arc of a circle to radius r extending from arc-chord $2k$ in one direction to arc-chord $2k$ in the contrary direction, and p the perpendicular from the origin of s on the direction of ds. For, calling n the element of the normal (drawn at the extremity of s) intercepted between two circular contours of radii r, $r+dr$, respectively, and remembering that the point of contact between these two circles is their centre of similitude, we have $\frac{n}{p}$ constant and consequently $= \frac{d\,(2r)}{2r} = \frac{dr}{r}$, so that $\Sigma\,(nds) = \frac{dr}{r}\Sigma\,(pds)$. Thus, in fact, if A is the area of a segment, to radius r, whose chord is $2k$, we have

$$\Sigma\,(pds) = 2\,(2\mathrm{A}) = 2\int_0^k \frac{(2\rho)^2\,d\rho}{\sqrt{(r^2-\rho^2)}} = 8\int_0^k \frac{\rho^2 d\rho}{\sqrt{(r^2-\rho^2)}};$$

and if we write $\rho = ru$, and then make $\frac{k}{r} = t$, and finally replace under the signs of summation u and t each by k, we obtain

$$\mathrm{F} - 2\pi^2 k^2 + \pi^2 k^4 = 32\pi k^4 \int_1^k \frac{dk}{k^3}\int_1^k \frac{dk}{k^3}\left\{\int_1^k \frac{k^2 dk}{\sqrt{(1-k^2)}} + \int_0^1 \frac{k^2 dk}{\sqrt{(1-k^2)}}\right\}$$

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$$= 4\pi k^4 \left\{ \frac{1}{k^4} \int_1^k \frac{k^2 dk}{\sqrt{(1-k^2)}} - \frac{2}{k^2} \int_1^k \frac{dk}{\sqrt{(1-k^2)}} + \int_1^k \frac{dk}{k^2\sqrt{(1-k^2)}} \right\} + \pi^2 (1-k^2)^2$$

$$= 2\pi \left\{ \sin^{-1} k - k\sqrt{(1-k^2)} - \tfrac{1}{2}\pi \right\} - 8\pi k^2 (\sin^{-1} k - \tfrac{1}{2}\pi) - 4\pi k^3 \sqrt{(1-k^2)} + \pi^2 (1-k^2)^2 ;$$

$$\therefore\ p = \frac{F}{\pi^2} = \frac{2}{\pi}(1-4k^2)\sin^{-1} k - \frac{2}{\pi}(k+2k^3)\sqrt{(1-k^2)} + 4k^2.$$

The same result might, of course, have been obtained, but less elegantly and less concisely, by writing at once in place of A its value as a function of k and r.

2. *For the sphere.* Following precisely the same method as in (α), (β), (γ) of the preceding, but of course substituting spherical shells for rings, we have

$$F - 2\left(\tfrac{4}{3}\pi\right)\left(\tfrac{4}{3}\pi k^3\right) + \left(\tfrac{4}{3}\pi\right)^2 k^6 = \int_k^1 2\,(4\pi r^2 dr) \int_k^r \frac{dr}{r}\ (B),$$

where $B = \Sigma\,(pdS)$, dS being an element of a spherical surface swept out by the revolution of the arc of the circle of radius r, cut off by the chord $2k$, about a diameter passing through an extremity of the arc. Thus B is three times the volume swept out by A in this revolution; that is, h being the versed-sine of the arc, $B = \pi h^2 r = \dfrac{4\pi k^4}{r}$; hence the probability is

$$p = 2k^3 - k^6 + 18k^4 \int_k^1 r^2 dr \int_k^r \frac{dr}{r^3} = 8k^3 - 9k^4 + 2k^6.$$

NOTE.—The solution here given, although it may appear unnecessarily elaborate as regards the particular problem to be resolved, is of importance as illustrating a general method developed by the writer for dealing with questions of this sort—a method which subjectively has taken its rise out of the admirable observation made use of by Mr. Woolhouse in his solution of the problem (Quest. 1333) for determining the probability of a triangle joining three points taken at random within a sphere being acute. Mr. Woolhouse, and Mr. Crofton after him, (*Educational Times* for Jan. 1863 and Oct. 1865, or *Reprint*, Vol. IV., pp. 61, 81,) have shown that one point of the group of three may be considered as fixed in the circumference without affecting the probability in question. See a brief notice of my paper read before the British Association at Birmingham, in the *Athenæum* and *Reader* of Oct. 7th, 1865, [also a notice, unfortunately still briefer for want of space, on p. 179 of the *Educational Times* for Nov., 1865]. The reductions in the order of summations therein referred to can be increased to *three* degrees for plane rectilinear, and to *four* degrees for solid polyhedral, contours.

[If we apply the foregoing method to find the probability ($1-p$, or q) that the distance between the two points is *greater* than $2k$, we need only attend to the process in (γ) and (δ), no other terms entering into our investigation; the complementary probability in question then is, for the circle,

$$q = \frac{1}{\pi^2} \int_k^1 4\pi r\, dr \int_k^r \frac{dr}{r} \int_k^r \frac{8\rho^2 d\rho}{\sqrt{(r^2-\rho^2)}} = \frac{4k^4}{\pi} \left(\int_1^{\frac{1}{k^2}} dx \right)^3 \frac{1}{x^2\sqrt{(x-1)}};$$

the latter remarkably simple result being obtained from the former by putting

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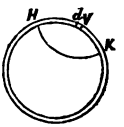
first $\rho = ru$, then $r = \frac{k}{t}$, and finally $x = \frac{1}{t^2}$. Reducing this to a series of (three) single integrals, by the aid of a known and elegant theorem for effecting such reduction of an integral of the nth order (see Boole's *Differential Equations*, 2nd ed., p. 398), the result is readily seen to be identical with that given above.]

II. *Solution by the* PROPOSER.

Let r be the given radius, a the length of the given line; and let us consider the problem, first, for the *sphere*.

If p be the probability of any event, F the number of favourable cases, W the whole number of cases; we have $p = \frac{F}{W}$; so that if F and W are determined, p is known.

We may represent the number of points within a given space, by the space itself; hence if S, S′ be two spaces, the number of pairs of points given by taking one point in each is SS′; and the number of pairs given by points within the same space S is $\frac{1}{2}S^2$. In the present question, $W = \frac{8}{3}\pi r^6$. To determine F, let r become $r + dr$; and, taking an infinitely small element dV of the spherical shell so formed as centre, draw a sphere of radius a cutting the other; the *increase* in the number of favourable cases given by taking one point in the element dV is (lune HK) $\times dV$; hence the whole increase is



$$dF = (\text{lune HK}) \times (\text{volume of spherical shell}).$$

Now lune $HK = \frac{2\pi}{3}a^3 - \frac{\pi}{4}\frac{a^4}{r}$, $\therefore dF = \left(\frac{2\pi}{3}a^3 - \frac{\pi}{4}\frac{a^4}{r}\right) 4\pi r^2 dr$;

$$\therefore F = 4\pi^2 a^3 \int \left(\frac{2}{3} - \frac{a}{4r}\right) r^2 dr = 4\pi^2 a^3 \left(\frac{2}{9}r^3 - \frac{a}{8}r^2\right) + C;$$

therefore $$p = \frac{F}{W} = \frac{a^3}{r^3} - \frac{9}{16}\frac{a^4}{r^4} + \frac{C'}{r^6}.$$

To determine the arbitrary constant C′, we observe that if $a = 2r$, $p = 1$, whence $C' = \frac{1}{32}a^6$; therefore the required probability is

$$p = \frac{a^3}{r^3} - \frac{9}{16}\frac{a^4}{r^4} + \frac{1}{32}\frac{a^6}{r^6}.$$

If the distance of the points is to be less than the *radius* of the *sphere* (or $a = r$), we have $p = \frac{15}{32}$.

For a *circle* the integration is rather more difficult; I find the result to be

$$p = \frac{a^2}{r^2} + \frac{2}{\pi}\left(1 - \frac{a^2}{r^2}\right) \sin^{-1}\frac{a}{2r} - \frac{a}{4\pi r}\left(2 + \frac{a^2}{r^2}\right)\left(4 - \frac{a^2}{r^2}\right)^{\frac{1}{2}}.$$

In this case if $a = r$, we have $p = 1 - \frac{3\sqrt{3}}{4\pi}$.

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Note on Question 1333; by W. S. B. Woolhouse, F.R.A.S.

The following summary statement is made at the suggestion of the Mathematical Editor of the *Educational Times*, and also at the request of Professor Sylvester, who has latterly become much interested in the general principles which bear upon the subject.

This problem was first proposed by me amongst the mathematical questions of the *Lady's and Gentleman's Diary* for 1861, and an able answer to it by Mr. Stephen Watson was given in the Diary for 1862. Another correct and satisfactory answer was also sent by Mr. C. H. Brooks, but it was too elaborate for insertion in that publication.

The solution printed in the *Educational Times* for January, 1863, was literally copied from my own manuscript investigation, drawn up some years before the question was proposed. In working out this solution, the principle of limiting one of the points to the surface of the sphere was really considered to be self-evident, and a sort of preliminary deduction [added at the end of this Note] was given more particularly for the purpose of adverting to the remarkable peculiarity that the acute triangles occupying the sphere and shell in the manner stated are not proportionally partitioned in the respective spaces. The determination of the probabilities p'', p''' there defined, will be found to be a curious subject of inquiry, when the shell is supposed to have a finite thickness.

So far as I am concerned as the originator of this question, I do not see much to claim in the suggestive idea of limiting one of the points to the surface; but whatever merit may be attached to it in the estimation of others, must be fully shared by Mr. Brooks, who, in his solution sent to the Diary, expressly adopts the same principle. Being myself at first under the impression that this simplifying expedient was essential to the investigation, I was somewhat pleased with the ingenuity shown by Mr. Watson, whose solution, printed in the Diary, is in this respect quite independent and unique.

Considering the problem in its more general form, whatever be the boundary of the figure, whether regular or irregular, it is obvious that one of the points may be restricted to the outer surface under certain conditions. Thus, for example, if within any proposed volume a point be taken as a centre of similitude, and the figure be conceived to receive a proportional increment from the centre in every direction, or if the augmented figure be only similar to the original one, whether similarly situated or not, it will follow that the probability will be the same if one of the points be restricted to the surface, provided that there be assigned to its existence a probability measured by the proportional increment of volume which appertains to the immediate vicinity of the point in question. In the case of the sphere, the point may not merely be restricted to the surface, but, on account of the general symmetry, it may be made a fixed point.

The most difficult phase of this question, and by far the most interesting, is that in which the three points are taken within the volume of a given parallelepiped, and to this no solution has yet been given. But my own investigation of the analogous case of a rectangle has already appeared in the Reprint of "Mathematical Questions with their Solutions, from the Educational Times," vol. i. p. 71.

In conclusion I may add that the principle by which a point is limited to a boundary may be stated in still more general terms as follows:—

If a figure be subjected to a small variation, in such a manner that two accumulative symmetrical functions of any kind, depending upon a given

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number of arbitrary points, shall have to each other the same ratio in the varied as in the primitive figure, that ratio will retain the same value when one of the points is restricted, positively or negatively, to the increment or decrement of the figure, as the case may be.

As regards the proof, it is sufficient to indicate that it follows from the simple and obvious analytical deduction, that if $\frac{Q+\delta Q}{R+\delta R}=\frac{Q}{R}$, however small the increments $\delta Q, \delta R$ be taken, then will $\frac{dQ}{dR}=\frac{Q}{R}$; and this applies equally to any form of enunciation.

[The "preliminary deduction" referred to above is as follows:—

"Let V denote the volume of a sphere, the number of available points comprised within it being proportionally represented by V. Then the number of triangles that can be formed by taking points three and three, will be $\frac{V^3}{2.3}$. Suppose V to receive a finite increment, and to become $V+\delta V$; then the total number of triangles that can be formed in the increased volume will be

$$\frac{(V+\delta V)^3}{2.3}=\frac{V^3}{2.3}+\frac{V^2}{2}.\delta V+V\frac{\delta V^2}{2}+\frac{\delta V^3}{2.3}, \text{ or } N=n_0+n_1+n_2+n_3\ldots.(\alpha).$$

Here the first term n_0 is the number of triangles wholly contained in V; n_1 is the number of those which have two angular points in V and one in δV; n_2 is the number of those which have all three points within the spherical shell δV. Let p denote the required probability (of an acute triangle), then the number of *acute* triangles contained amongst N will be $pN=pn_0+pn_1+pn_2+pn_3$; but it is a peculiar feature of this question that the acute triangles are not thus locally partitioned as in (α). If p', p'', p''' be the respective probabilities under the several partitions, the values collected under those partitions will be $pN=pn_0+p'n_1+p''n_2+p'''n_3$, and p', p'', p''' will, in fact, be functions of $\frac{\delta V}{V}$.

Equating the two values of pN thus represented, we must have

$$0=(p'-p)+(p''-p)\frac{n_2}{n_1}+(p'''-p)\frac{n_3}{n_1}=(p'-p)+(p''-p)\frac{\delta V}{V}+\tfrac{1}{2}(p'''-p)\frac{\delta V^2}{V^2}.$$

If we now conceive the finite increment δV to be evanescent, this equation will become $p_0'-p=0$, or $p=p_0'$; that is, the required probability is the same as that which obtains when one of the points is limited to the surface of the given sphere."]

1790. (Proposed by Professor SYLVESTER.)—(1.) If a set of six points be respectively represented by the six permutations of $a:b:c$, show that they lie in a conic, and write down its equation.

(2.) *Hence* prove that if AB, BC, CA be three real lines containing the nine points of inflexion of a cubic curve having an oval, the pairs of tangents drawn to the oval from A, B, C will meet it in six points lying in a conic.

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I. *Solution by* Professor CAYLEY.

1. That the six points,

$$1 = (a, b, c), \quad 2 = (b, c, a), \quad 3 = (c, a, b),$$
$$4 = (a, c, b), \quad 5 = (b, a, c), \quad 6 = (c, b, a),$$

are situate on a conic, appears at once by writing down its equation: viz.,

$$(bc + ca + ab)(x^2 + y^2 + z^2) - (a^2 + b^2 + c^2)(yz + zx + xy) = 0,$$

which is satisfied by the coordinates of each of the six points.

2. It is interesting to remark that the six points on the conic form, not a general inscribed hexagon, but a hexagon such as is mentioned in Prob. 1512 (*Reprint*, Vol. II., p. 51), viz., one in which the three diagonals pass respectively through the Pascalian points (intersections of opposite sides): in fact, in the hexagon 143526, forming the equations of the sides and diagonals, these are

$$14.\ x(b+c) - ay - az = 0, \qquad 25.\ x(c+a) - by - bz = 0,$$
$$15.\ -cx - cy + (a+b)z = 0, \qquad 26.\ -ax - ay + (b+c)z = 0,$$
$$16.\ -bx + (c+a)y - bz = 0, \qquad 24.\ -cx + (a+b)y - cz = 0,$$
$$36.\ x(a+b) - cy - cz = 0,$$
$$34.\ -bx - by + (c+a)z = 0,$$
$$35.\ -ax + (b+c)y - az = 0;$$

so that the lines 14, 25, 36 meet in the point $x = 0,\ y + z = 0$,
" 16, 24, 35 " $y = 0,\ z + x = 0$,
" 15, 26, 34 " $z = 0,\ x + y = 0$.

3. It is further to be remarked that the six points lie on the cubic curve

$$\frac{x^3 + y^3 + z^3}{a^3 + b^3 + c^3} - \frac{xyz}{abc} = 0,$$

and are consequently the six points of intersection of this cubic by the above mentioned conic.

4. The points $(x = 0,\ y + z = 0)$, $(y = 0,\ z + x = 0)$, $(z = 0,\ x + y = 0)$ are the three real inflexions of the cubic; hence, attending only to the cubic, and starting from the arbitrary point (a, b, c) on this curve, it appears by what precedes, that we may, by means of the three real inflexions of the cubic, construct the system of six points, (the construction is, in fact, identical with that given in my Solution of Problem 1744, *Reprint*, Vol. IV., pp. 32—37, the six points being six out of the therein mentioned eighteen points); and it further appears, that these six points lie on a conic.

5. As regards the second part of the proposed Problem, consider the cubic curve $x^3 + y^3 + z^3 + 6lxyz = 0$; the three real lines containing the nine points of inflexion are the lines $x = 0$, $y = 0$, $z = 0$; and the points A, B, C are therefore $(y = 0,\ z = 0)$, $(z = 0,\ x = 0)$, $(x = 0,\ y = 0)$ respectively. From each of these points we may draw to the curve six tangents, and we have thus on the curve eighteen points, which are a particular case of the system in the Solution of Prob. 1744. Or if from each of the points we draw two properly selected tangents, (when the cubic has an oval these may be the two tangents to the oval,) then we obtain a system of six points, (part of the system of eighteen points); viz., the coordinates of the six

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points are of the form (a, b, c), (b, c, a), (c, a, b), (a, c, b), (b, a, c), (c, b, a) and therefore the six points are in a conic.

6. To verify this, if we take $y = \theta x$ for the equation of a tangent from the point $(x = 0, y = 0)$, the equation $(1 + \theta^3) x^3 + 6l\theta x^2 z + z^3 = 0$ must have a pair of equal roots, giving for θ the equation $(1 + \theta^3)^2 + 32l^3\theta^3 = 0$; and we then find $z = -\dfrac{1+\theta^3}{4l\theta} x$, that is, θ being determined by the foregoing equation, the coordinates of the point of contact are $x : y : z = 1 : \theta : -\dfrac{1+\theta^3}{4l\theta}$. The roots of the equation in θ are of the form θ_1, θ_2, θ_3, $\dfrac{1}{\theta_1}$, $\dfrac{1}{\theta_2}$, $\dfrac{1}{\theta_3}$; and assuming that the curve has an oval, there are two real roots θ_1, $\dfrac{1}{\theta_1}$. Hence, writing $x : y : z = 1 : \theta_1 : -\dfrac{1+\theta_1^3}{4l\theta_1} = a : b : c$, the substitution $\dfrac{1}{\theta_1}$ for θ, gives $x : y : z = b : a : c$, that is, the coordinates of the points of contact of the tangents to the oval, from the point $(x = 0, y = 0)$ are (a, b, c) and (b, a, c) respectively; and writing successively (y, z, x) and (z, x, y) in place of $(x. y, z)$, the coordinates for the tangents from $(y = 0, z = 0)$ are (b, c, a), (c, b, a); and those for the tangents from $(z = 0, x = 0)$ are (c, a, b) and (a, c, b); so that the coordinates of the six points of contact are a system of the form in question.

II. *Solution by the* Proposer.

(1.) This needs no proof: it is enough to *write down* $\Sigma ab \Sigma x^2 - \Sigma a^2 \Sigma yz = 0$.
(2.) Writing the equation to the cubic under the form $x^3 + y^3 + z^3 + 6lxyz = 0$, where l is real, its points of contact with the 6 tangents from $(x=0, y=0)$ will be found by combining the above with $z^2 + 2lxy = 0$, which gives

$$\frac{x^3}{z^3} + \frac{y^3}{z^3} = 2, \quad \frac{x^3}{z^3} \cdot \frac{y^3}{z^3} = -\frac{1}{8l^3}.$$

Hence $\dfrac{x^3}{z^3}$, $\dfrac{y^3}{z^3}$ are the roots, say ρ, ρ' of the equation $\rho^2 - 2\rho - \dfrac{1}{8l^3} = 0$;

and ρ, ρ' must be real, or there could not be an oval; for the oval (when there is one) always lies entirely within or entirely without the triangle ABC. The six points of contact with the *cubic* arise from combining the three values of $\sqrt[3]{\rho}$ with the three values of $\sqrt[3]{\rho'}$, subject to the condition that $\sqrt[3]{\rho} \cdot \sqrt[3]{\rho'} = -\dfrac{1}{2l}$. Thus, using $\sqrt[3]{\rho}$, $\sqrt[3]{\rho'}$ to denote the real values of the cube roots, the two points of contact with the oval of the tangent from $(x=0, y=0)$ will have $1 : \sqrt[3]{\rho} : \sqrt[3]{\rho'}$ and $1 : \sqrt[3]{\rho'} : \sqrt[3]{\rho}$ for their x, y, z coordinates respectively. And proceeding in like manner with the other two pairs of tangents, we see that the six permutations of $1 : \sqrt[3]{\rho} : \sqrt[3]{\rho'}$ represent the x, y, z coordinates of the six points of contact with the oval; these latter therefore, by (1), lie in a conic.

[We have also received a solution of the Question from Professor Cremona, agreeing in principle with that by Professor Cayley.]

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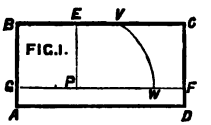
1321. (Proposed by the EDITOR.)—If two marbles are thrown at random on the floor of a rectangular room, what is the chance that they will rest at a distance apart less than a given distance ?

Solution by M. W. CROFTON, B.A.

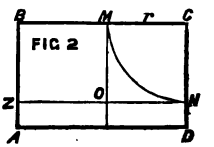


Let a be the length (AD) of the room ABCD, b its breadth (AB), and r the given distance. We shall get all the possible cases by making one point move over the whole surface of the rectangle (Fig. 1), and, in each of its positions (P), taking the other point at random in the rectangle GBCF; and it is easy to see that this is equivalent to taking the other point at random in the rectangle PECF, and doubling the result. To find, then, the number of favourable cases, we shall take P in any position, and draw a circle with P as centre and r as radius; then the arc VW of this circle, which falls inside the rectangle PECF, separates the favourable positions of the second point from the unfavourable; and if dS is the element of the surface at P, the measure (F) of the favourable cases will be

$$F = 2\iint (dS \,.\, EVWP).$$

We shall find it more convenient, however, to find the differential of F with regard to r; if r receive an increment dr, the corresponding increment of F, for the particular position of P, will be $2dS$. arc VW . dr,

therefore
$$\frac{dF}{dr} = 2\iint \text{arc VW} \,.\, dS.$$



We have then only to determine the value of the arc VW in terms of the coordinates of P for each different position of P.

1. *In the first case, when r is less than a or b.* Form (Fig. 2) a square CO whose side is r, it will be easily seen that

if P is in AO, arc $\text{VW} = r\frac{\pi}{2}$,

if P is in BO, arc $\text{VW} = r \sin^{-1}\frac{y}{r}$,

if P is in DO, arc $\text{VW} = r \sin^{-1}\frac{x}{r}$

if P is in the quadrant CMN, arc VW = 0,

if P is in the mixtilinear space MON, arc $\text{VW} = r\left(\sin^{-1}\frac{y}{r} - \cos^{-1}\frac{x}{r}\right)$.

N.B.—In each of the 4 rectangles in the figure the axes of reference taken are its two sides which belong to the original rectangle; *e.g.*, when P is in BO, the axes are BM, BZ.

Hence
$$\frac{1}{2}\frac{dF}{dr} = r\int_0^{a-r}\int_0^{b-r}\frac{\pi}{2}\,dxdy + r\int_0^{a-r}dx\int_0^{r}dy\,\sin^{-1}\frac{y}{r}$$

$$+ r\int_0^{b-r}dy\int_0^{r}dx\,\sin^{-1}\frac{x}{r} + r\int_0^{r}dx\int_{(r^2-x^2)^{\frac{1}{2}}}^{r}dy\left(\sin^{-1}\frac{y}{r} - \cos^{-1}\frac{x}{r}\right)$$

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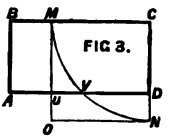
Performing the integrations, which are very easy, we find

$$\frac{1}{2}\frac{dF}{dr} = \frac{\pi}{2}abr - (a+b)r^2 + \frac{1}{2}r^3, \quad \therefore \ F = \frac{\pi}{2}abr^2 - \frac{2}{3}(a+b)r^3 + \frac{1}{4}r^4.$$

No constant is added, as F should vanish with r. Now the measure (W) of the whole number of cases is $\frac{1}{2}a^2b^2$. Hence the probability required is

$$p = \frac{F}{W} = \frac{r^2}{a^2b^2}\left\{\pi ab - \frac{4}{3}(a+b)r + \frac{1}{2}r^2\right\}.$$



2. *When r is less than a and greater than b.* Here we have to integrate (Fig. 3) over the spaces BU and MUV; then

$$\frac{1}{2}\frac{dF}{dr} = \int_0^{a-r} dx \int_0^b dy \sin^{-1}\frac{y}{r} + r\int_0^b dy \int_{(r^2-y^2)^{\frac{1}{2}}}^{r} dx \left(\sin^{-1}\frac{y}{r} - \cos^{-1}\frac{x}{r}\right)$$

$$= ar\left\{b\sin^{-1}\frac{b}{r} + (r^2-b^2)^{\frac{1}{2}} - r\right\} - \frac{b^2}{2}r;$$

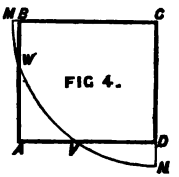
$$\therefore \frac{1}{2}F = ab\int \sin^{-1}\frac{b}{r}\,rdr + a\int r(r^2-b^2)^{\frac{1}{2}}\,dr - \frac{ar^3}{3} - \frac{b^2r^2}{4},$$

$$\therefore F = abr^2\sin^{-1}\frac{b}{r} + \frac{a}{3}(2r^2+b^2)(r^2-b^2)^{\frac{1}{2}} - \frac{2ar^3}{3} - \frac{b^2r^2}{2} + C.$$

The constant C is found by considering that if $r = b$, the value of F is the same as in case 1, hence we find that $C = \frac{1}{12}b^4$;

$$\therefore p = \frac{2r^2}{ab}\sin^{-1}\frac{b}{r} + \frac{2}{3}\cdot\frac{2r^2+b^2}{ab^2}(r^2-b^2)^{\frac{1}{2}} - \frac{4}{3}\cdot\frac{r^3}{ab^2} - \frac{r^2}{a^2} + \frac{1}{6}\cdot\frac{b^2}{a^2}.$$

3. *When r is greater than a.* Here we have to integrate (Fig. 4.) over the space VAW; then



$$\frac{1}{2}\cdot\frac{dF}{dr} = r\int_{(r^2-a^2)^{\frac{1}{2}}}^{b} dy \int_{(r^2-y^2)^{\frac{1}{2}}}^{a} dx \left(\sin^{-1}\frac{y}{r} - \cos^{-1}\frac{x}{r}\right)$$

$$= r\left\{ab\left(\sin^{-1}\frac{b}{r} - \cos^{-1}\frac{a}{r}\right) + a(r^2-b^2)^{\frac{1}{2}} + b(r^2-a^2)^{\frac{1}{2}} - \frac{a^2}{2} - \frac{b^2}{2} - \frac{r^2}{2}\right\}$$

$$\therefore \frac{1}{2}F = ab\int \sin^{-1}\frac{b}{r}\,rdr + ab\int \sin^{-1}\frac{a}{r}\,rdr - \frac{\pi}{2}ab\int rdr$$

$$+ a\int (r^2-b^2)^{\frac{1}{2}}\,rdr + b\int (r^2-a^2)^{\frac{1}{2}}\,rdr - \frac{a^2r^2}{4} - \frac{b^2r^2}{4} - \frac{r^4}{8};$$

$$\therefore F = abr^2\left(\sin^{-1}\frac{a}{r} + \sin^{-1}\frac{b}{r}\right) - \frac{\pi}{2}abr^2 + \frac{a}{3}(2r^2+b^2)(r^2-b^2)^{\frac{1}{2}}$$

$$+ \frac{b}{3}(2r^2+a^2)(r^2-a^2)^{\frac{1}{2}} - \frac{r^2}{2}(a^2+b^2) - \frac{r^4}{4} + C.$$

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To determine C, let $r = a$, then the value of F should be the same as in case (2), hence we find that $C = \frac{1}{12}(a^4 + b^4)$;

$$\therefore p = \frac{2r^2}{ab}\left(\sin^{-1}\frac{a}{r} + \sin^{-1}\frac{b}{r}\right) - \frac{\pi r^2}{ab} + \frac{2}{3}\cdot\frac{2r^2+a^2}{a^2b}(r^2-a^2)^{\frac{1}{2}}$$
$$+ \frac{2}{3}\cdot\frac{2r^2+b^2}{ab^2}(r^2-b^2)^{\frac{1}{2}} - r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) - \frac{r^4}{2a^2b^2} + \frac{1}{6}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right).$$

NOTE.—If we call the function which expresses the probability in the first case, $F(a, b, r)$; in the second, $f(a, b, r)$; and in the third, $\phi(a, b, r)$; we shall have the following remarkable relation between the forms of these functions,

$$F(a, b, r) + \phi(a, b, r) = f(a, b, r) + f(b, a, r);$$

evidently pointing to some law in the analytical theory of probability, connecting the functions in problems which give rise to several cases, and which would be well worthy of the attention of mathematicians. To take a different class of problem, let us suppose two bags containing m and n balls respectively ($m > n$): a person draws an arbitrary number of balls (0 being admissible) from each; to find the chance of his drawing an assigned number (x) in all. It will be easily found that if x is between 0 and (n), the chance is $F(m, n, x) = \frac{x+1}{(m+1)(n+1)}$; if x is between (n) and (m), $f(m, n, x) = \frac{1}{m+1}$; if x is between (m) and ($m+n$), $\phi(m, n, x) = \frac{m+n-x+1}{(m+1)(n+1)}$.

Here also $(m, n, x) + \phi(m, n, x) = f(m, n, x) + f(n, m, x)$,

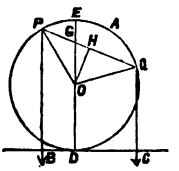
and there can be no doubt that analogous results will be found to hold in similar cases.

[A solution by Mr. WOOLHOUSE will be found in the *Educational Times* for March, 1863.]

1765. (Proposed by EXHUMATUS.)—Two weights are attached by short strings of given length to the surface of a cylinder; determine the position of equilibrium of the cylinder when resting on a horizontal plane, the weights resting on the surface of the cylinder.

Solution by E. L. ROSOLIS.

Let O be the centre of the cylinder, P and Q the weights, A the point to which the strings AP and AQ are attached; join PQ, OP, and OQ; draw PB, QC, and EOD perpendicular to the horizontal plane BC, and OH perpendicular to PQ. Let θ = inclination of PQ to the horizon = $\angle EOH$, r = radius of cylinder, $AP = m$, $AQ = n$, 2α = angle whose circular measure is $\frac{m+n}{r}$ = $\angle POQ$. In the position of equilibrium the



centre of gravity (G) of P and Q must be in the vertical line DE, therefore

$$\frac{P}{Q} = \frac{QG}{PG} = \frac{CD}{BD} = \frac{\sin(\alpha+\theta)}{\sin(\alpha-\theta)}, \text{ whence } \tan\theta = \frac{P-Q}{P+Q}\tan\alpha.$$

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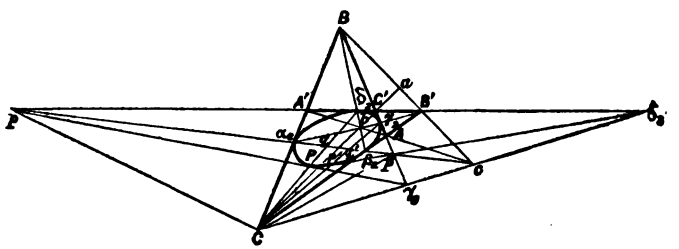
1789. (Proposed by Professor WHITWORTH.) — Let *abc* be the triangle formed by the three diagonals AA′, BB′, CC′ of a quadrilateral; and let A*a*, B′*b*, C′*c* meet A′BC in a_1, a_2, a_3; A′*a*, B*b*, C′*c* meet AB′C in β_1, β_2, β_3; A′*a*, B′*b*, C*c*, meet ABC′ in γ_1, γ_2, γ_3; A*a*, B*b*, C*c*, meet A′B′C′ in δ_1, δ_2, δ_3; then a_2, β_2, γ_2, δ_2 are the points of contact of the "*Critical* Inscribed Conic" of the quadrilateral CA′C′A (this may be taken as the *Definition* of the "*Critical* Inscribed Conic"), and Professor Cayley has given in Quest. 1751 (*Reprint*, Vol. IV., p. 38), a construction for finding four other points on the conic and the tangents thereat.

It is required to show that, by the ruler alone, *eight* other points on this conic, and the tangents thereat, may be found by the following

Construction.—Let $a_2\beta_2$ meet C′A′ in p; join $p\gamma_2$ cutting Cδ_2 in P; then P is a point on the curve, and Pp the tangent at P. Similarly, *seven* other points may be found, and the tangents constructed thereat.

Solution by J. DALE.



From the quadrilateral CA′B′*c* Bδ_2 it is evident that C*c*$\gamma_2\delta_2$ is a harmonic range; therefore, as $a_2\beta_2$ passes through *c*, C*q*Pδ_2 is also a harmonic range; and as C is the polar of $a_2\beta_2$, and δ_2 one of the points in which C*q* meets the curve, it follows that P is the other point; and as p is the pole of Cδ_2, and $p\delta_2$ *one* of the tangents from p, therefore pP is the *other*. Again, let $a_2\beta_2$ produced meet C′A in p'; join $p'\delta_2$ cutting Cγ_2 in P′; then p'P′ touches the conic in P′. The proof is the same as before, since the line Cγ_2 is harmonically divided in q', P′. So also may other six points be found on the lines C′a_2, C′β_2; Aa_2, Aδ_2; A′β_2, A′γ_2; respectively.

II. *Solution by the* PROPOSER.

Take the given quadrilateral as quadrilateral of reference for that system of quadrilinear coordinates where the coordinates of every point are subject to the identity $\alpha+\beta+\gamma+\delta \equiv 0$ .....(*i*).

Then (as I have shown in my paper on Quadrilinear Coordinates, *Messenger of Mathematics*, Vol. III., p. 174, Art. 32.) the critical conic inscribed in the quadrilateral CAC′A′ is represented by either of the equivalent equations

$$(\alpha-\gamma)^2 = 4\beta\delta, \text{ or } (\beta-\delta)^2 = 4\alpha\gamma \text{ .....(ii)},$$

or by any equation formed by combining either of these with the identity (*i*). Now since $\alpha-\gamma \equiv 2\alpha+\beta+\delta$, the first equation to the conic may be written

$$(2\alpha+\beta+\delta)^2 = 4\beta\delta, \text{ or } (2\alpha+\beta)^2+\delta\,(4\alpha-2\beta+\delta) = 0 \text{ .....(iii)},$$

which shows that $(\delta = 0)$ and $(4\alpha-2\beta+\delta = 0)$ represent tangents at the

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extremities of the chord ($2\alpha+\beta=0$), or $C\delta_2$. Now p is given by $2\alpha-\beta+\delta=0$ (or $a_2\beta_2$) and $\delta=0$ (or $C'A'$), and therefore lies on the line ($4\alpha-2\beta+\delta=0$). And this equation is also satisfied when $\alpha-\beta=0$ and $\gamma=0$, i.e. at γ_3. Hence $4\alpha-2\beta+\delta=0$ represents the line $p\gamma_3$, which therefore touches the conic at its point of intersection with $C\delta_2$.

On Triangular Symmetry. By W. K. Clifford.

We make the properties of a conic intuitive by studying it under the form of a circle; or those of a quadrilateral, by studying it under the form of a square. This simplification depends upon the projective property of a right angle, viz., that it divides harmonically the chord at infinity of a circle. By means of this property we interpret as general, propositions whose truth we see intuitively through the symmetry of the figure. This kind of symmetry (that of a circle or square) I call the *symmetry of the right angle*, or *rectangular symmetry*.

From the symmetry of two lines we ascend immediately to the symmetry of three lines, or of the equilateral triangle. This is exemplified in the Rhombohedral System of Crystals, just as Rectangular Symmetry is exemplified in the Pyramidal System. I want in this note to call attention to the uses of Triangular Symmetry in presenting general propositions under an intuitive form.

The projective property of an equilateral triangle is this: *it determines on the line at infinity a point-cubic whose Hessian is the circular points.* Given four lines, we may project one of them to infinity so that the other three shall form an equilateral triangle; for we have only to construct the Hessian of the point-cubic determined upon that one by the other three, and then project this Hessian into the circular points. Similar problems are

B. To project at once a conic and triangle into a circle and an equilateral triangle.

C. To project at once two conics and a triangle into two rectangular hyperbolæ and an equilateral triangle.

D. To project at once two triangles into equilateral triangles.

In each of these we have the problem of finding the line which has to be projected to infinity; this problem admits, in the three cases respectively, of 43, 76, and 108 solutions. The *triangle* might of course be replaced by a *cubic*, to be so projected that its asymptotes should form an equilateral triangle; but this case is not particularly interesting. Every cubic may be projected into a perfectly symmetrical form in this way;—the three real inflexions of the cubic lie on a certain straight line, and determine a point-cubic upon it; let the Hessian of this point-cubic be projected into the circular points at infinity. Then the cubic is symmetrically situated in respect of an equilateral asymptotic triangle.

The form in Fig. 1 may be called *inscribed*, that in Fig. 2 *escribed*. The inscribed cubic may have no oval nor double point, and then will lie entirely without the triangle.

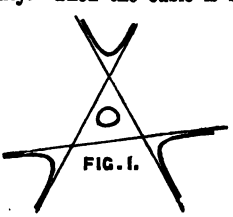
FIG. 1.

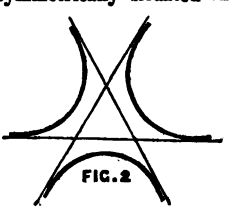
FIG. 2

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Cubics with a crunode cannot be thus symmetrized; but their reciprocals can. In fact, every three-cusped quartic can be projected into a hypocycloid of three branches. For, any four points can be projected into any other four points; if then the three cusps and the intersection of the cuspidal tangents be projected respectively into the vertices and centre of an equilateral triangle, the thing is done. This proof is Professor CAYLEY'S. We learn in this way that, in any three-cusped quartic, the cuspidal tangents and the lines joining the cusps determine on the double tangent two point-cubics, whose common Hessian is the points of contact of that tangent.

Trinodal quartics have four double tangents, which are all real as lines when the nodes are real, but one is ideal (I borrow this convenient expression from Poncelet) or has imaginary contacts. If the three pairs of nodal tangents divide the ideal double tangent in the same anharmonic ratio, the quartic can be projected into a hypotrochoid, the contacts of the ideal tangent being then the circular points at infinity. The curve has many remarkable properties, which can be recognised at once from the symmetry of the projected figure.

Higher orders of symmetry are special cases. Thus quartic symmetry (as of a regular octagon) requires that the point-quartic shall be an harmonic system, so that its cubinvariant vanishes. Quintic symmetry (regular pentagon) requires a point-quintic of the form ax^5+fy^5. The conditions that the quintic may be reduced to this form are that the invariants K of the eighth degree and L of the twelfth shall separately vanish (see Professor SYLVESTER'S admirable *Trilogy*, *Phil. Trans.* 1864, p. 619).

The cube and sphere are examples of the symmetry of the *cubangle*, whose projective property is that it determines on the plane at infinity a conjugate triad of the imaginary circle. One is naturally led to seek for the projective property of a regular tetrahedron. It determines on the plane at infinity four straight lines, x, y, z, w; and if we assume the identical relation

$$x+y+z+w \equiv 0,$$ then the equation of the imaginary circle is

$x^2+y^2+z^2+w^2=0$, as is easily shown by the consideration that each face is an equilateral triangle. Many interesting properties of this conic will be found proved in the solution to Question 1690.

1741. (Proposed by J. GRIFFITHS, M.A.)—Let P be the point of intersection of the perpendiculars, and G the centre of gravity, of any triangle ABC; prove that the nine-point circles of all the triangles that can be formed from the points A, B, C, P, G, together with the maximum ellipse that can be inscribed in the triangle ABC, all pass through a common point, viz., the centre of the equilateral hyperbola passing through A, B, C, P, G.

Solution by J. DALE; E. MCCORMICK; *and others.*

The equilateral hyperbola passing through A, B, C, P, G, circumscribes each one of the ten triangles ABC, BPC, CPA, APB, BGC, CGA, AGB, PGA, PGB, PGC; and as the centre of an equilateral hyperbola circumscribing a triangle lies upon the nine-point circle of the triangle, it follows that the nine-point circles of the ten triangles pass through one point, viz., the centre of the equilateral hyperbola ABCPG.

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It now remains to be shown that the maximum inscribed ellipse also passes through this point. The equation of the hyperbola is

$$\frac{b^2-c^2}{a\alpha}+\frac{c^2-a^2}{b\beta}+\frac{a^2-b^2}{c\gamma}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the coordinates of the centre are connected by the equations

$$\frac{a\alpha}{(b^2-c^2)^2}=\frac{b\beta}{(c^2-a^2)^2}=\frac{c\gamma}{(a^2-b^2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equation of the maximum ellipse is

$$\sqrt{(a\alpha)}+\sqrt{(b\beta)}+\sqrt{(c\gamma)}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now it is evident that the values of α, β, γ, obtained from (2), satisfy (3), hence the centre of the equilateral hyperbola lies on the ellipse; therefore the centre of the hyperbola is a point common to each one of the ten nine-point circles, and to the maximum inscribed ellipse.

1675. (Proposed by W. K. CLIFFORD.)—If a triangle abc be the reciprocal of ABC in respect of a parabola whose parameter is $4m$; and if n_1, n_2, n_3 be the normals at the vertices of diameters through ABC; then

$$\frac{(\text{area of } abc)^2}{bc\,.\,ca\,.\,ab}=\frac{2m^2}{n_1n_2n_3}(\text{area of ABC}).$$

Solution by J. DALE; E. FITZGERALD; *and others.*

The equation of the parabola being $y^2=4mx$, let the coordinates o A, B, C be (h_1, k_1), (h_2, k_2), (h_3, k_3), respectively; then we have

$$2\,(\text{area of ABC})=\begin{vmatrix} h_1 & k_1 & 1 \\ h_2 & k_2 & 1 \\ h_3 & k_3 & 1 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

also $n_1^2=k_1^2+4m^2$, $n_2^2=k_2^2+4m^2$, $n_3^2=k_3^2+4m^2$; and the equations to bc, ca, ab (the polars of A, B, C) are, respectively,

$$k_1y=2m\,(x+h_1),\quad k_2y=2m\,(x+h_2),\quad k_3y=2m\,(x+h_3).$$

From these equations the coordinates of b, c are found to be

$$\frac{\begin{vmatrix} h_3 & k_3 \\ h_1 & k_1 \end{vmatrix}}{\begin{vmatrix} k_3 & 1 \\ k_1 & 1 \end{vmatrix}},\quad 2m\frac{\begin{vmatrix} h_3 & 1 \\ h_1 & 1 \end{vmatrix}}{\begin{vmatrix} k_3 & 1 \\ k_1 & 1 \end{vmatrix}};\quad \frac{\begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix}}{\begin{vmatrix} k_1 & 1 \\ k_2 & 1 \end{vmatrix}},\quad 2m\frac{\begin{vmatrix} h_1 & 1 \\ h_2 & 1 \end{vmatrix}}{\begin{vmatrix} h_1 & 1 \\ k_2 & 1 \end{vmatrix}};$$

and from these values of the coordinates, we find

$$(bc)^2=(k_1^2+4m^2)\begin{vmatrix} h_1 & k_1 & 1 \\ h_2 & k_2 & 1 \\ h_3 & k_3 & 1 \end{vmatrix}^2 \div \left(\begin{vmatrix} k_3 & 1 \\ k_1 & 1 \end{vmatrix}\begin{vmatrix} k_1 & 1 \\ k_2 & 1 \end{vmatrix}\right);$$

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with similar values for $(ca)^2$, $(ab)^2$; hence we obtain

$$\frac{bc}{n_1}\cdot\frac{ca}{n_2}\cdot\frac{ab}{n_3}=\frac{\begin{vmatrix} h_1 & k_1 & 1\\ h_2 & k_2 & 1\\ h_3 & k_3 & 1\end{vmatrix}^3}{\left(\begin{vmatrix} k_2 & 1\\ k_3 & 1\end{vmatrix}\begin{vmatrix} k_3 & 1\\ k_1 & 1\end{vmatrix}\begin{vmatrix} k_1 & 1\\ k_2 & 1\end{vmatrix}\right)^2} \quad \ldots\ldots\ldots (2.)$$

Moreover the area of the triangle abc is

$$m\begin{vmatrix} \begin{vmatrix} h_2 & k_2\end{vmatrix}\begin{vmatrix} h_2 & 1\\ h_3 & 1\end{vmatrix}\begin{vmatrix} k_2 & 1\\ k_3 & 1\end{vmatrix}\\ \begin{vmatrix} h_3 & k_3\end{vmatrix}\begin{vmatrix} h_3 & 1\\ h_1 & 1\end{vmatrix}\begin{vmatrix} k_3 & 1\\ k_1 & 1\end{vmatrix}\\ \begin{vmatrix} h_1 & k_1\end{vmatrix}\begin{vmatrix} k_1 & 1\\ k_2 & 1\end{vmatrix}\begin{vmatrix} k_1 & 1\\ k_2 & 1\end{vmatrix}\end{vmatrix}=\frac{m\begin{vmatrix} h_1 & k_1 & 1\\ h_2 & k_2 & 1\\ h_3 & k_3 & 1\end{vmatrix}^2}{\begin{vmatrix} k_2 & 1\\ k_3 & 1\end{vmatrix}\begin{vmatrix} k_3 & 1\\ k_1 & 1\end{vmatrix}\begin{vmatrix} k_1 & 1\\ k_2 & 1\end{vmatrix}} \quad (3).$$

The theorem in question follows at once from (1), (2), (3).

1672. (Proposed by J. TAYLOR.)—Give simple geometrical constructions for (1) inscribing the greatest triangle in an ellipse, and (2) circumscribing the least triangle about an ellipse.

Solution by the Rev. J. L. KITCHIN, M.A.

1. The greatest triangle in a circle is an equilateral triangle, and a side bisects the radius perpendicular to it. If the circle and triangle be projected orthogonally on a plane through the diameter parallel to a side, the circle becomes an ellipse, and the triangle is the greatest triangle in the ellipse. Its side also passes through the *middle* point of the semi-minor axis. If therefore we draw, through the middle point of the semi-minor axis, a line parallel to the major axis, and join the points in which it meets the curve with the further end of the minor axis, the triangle so formed is the *greatest* triangle which can be inscribed in the ellipse.

2. The *least* triangle about a circle is an equilateral triangle. If a circle be projected orthogonally on a plane through a diameter, the ellipse so formed bears the same relation to the circle, whence it is derived, as subsists between an ellipse and the circle on its major axis. Hence we may call the ellipse the *projection* of the circle, remembering that, when the projection is made, the plane is turned about the diameter until it coincides with the plane of the circle. Now, projecting the circle and its circumscribing triangle on a plane passing through a diameter parallel to one of the sides of the triangle, we obtain our ellipse and its least circumscribing triangle. The side parallel to the diameter is not changed in the projection. Hence the following construction:—Describe the circle on the major axis of the given ellipse, and about it an equilateral triangle. Through either end of the minor axis

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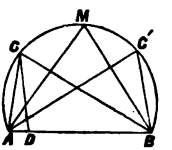
draw a tangent to the ellipse, bisected in that point, and equal to the side of the triangle. From the ends of this line draw tangents to the ellipse, and produce them to meet; the triangle so formed is the least triangle circumscribing the ellipse.

3. The Solution of Quest. 1781 may also be effected very simply by making use of orthogonal projection. For all that is necessary is to solve the problem in the case of the circle on the major axis of the ellipse. Then drawing ordinates from the angular points on the circle, we obtain, by their intersection with the ellipse, the angular points sought of the greatest triangle in the ellipse; for the latter is the *projection* of the former.

1807. (Proposed be R. TUCKER, M.A.)—From the ends of a diameter of a semicircle, two persons walk in random directions within the semicircle; find the chance of their paths intersecting inside the circle.

Solution by J. DALE; E. FITZGERALD; *and others.*



Suppose one walks along any line AC making an angle θ with AB, the path of the other, starting from B, will intersect AC so long as the angle ABC is less than $\frac{1}{2}\pi-\theta$; the chance of this is $(\frac{1}{2}\pi-\theta)\div\frac{1}{2}\pi$, and the mean value of this chance, supposing θ to vary from 0 to $\frac{1}{2}\pi$ is

$$\Sigma\left(\frac{\frac{1}{2}\pi-\theta}{\frac{1}{2}\pi}\cdot\frac{\Delta\theta}{\frac{1}{2}\pi}\right), \text{ or } \frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}\pi-\theta)\,d\theta = \tfrac{1}{2};$$

therefore, the chances of intersecting inside or outside the circle are equal.

[*Otherwise:* Let AMB be *any circular segment* (we have adapted Mr. Dale's figure to this form of the Problem), M the middle point of the arc, AC and AC′ any two lines equally inclined to AM; then, supposing all directions within the segment to be equally probable, and observing that the number of positions of AC′ is equal to those of AC, the measure (F) of the whole number of favourable cases will be $F = \frac{1}{2}$ (arc AC + arc AC′) × arc AB $= \frac{1}{2}$(arc AB)², and the measure W of the whole number of cases is W = (arc AB)²; hence the required probability is $p = \dfrac{F}{W} = \frac{1}{2}$.

Similar Problems may be proposed involving other curves. For instance, if AMB were an *ellipse*, AB (= 2*a*) the major axis, 2*b* the minor axis, and CD perpendicular to AB, we should have $a^2 : b^2 = AD \cdot DB : CD^2$ = cot CAD (or cot θ) : tan CBD; therefore $CBD = \tan^{-1}\left(\dfrac{b^2}{a^2}\cot\theta\right)$.

Hence the required probability will be expressed by the integral

$$\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\tan^{-1}\left(\frac{b^2}{a^2}\cot\theta\right)d\theta,$$

which, when $a = b$, becomes the integral given in Mr. Dale's Solution.]

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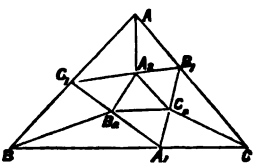
1726. (Proposed by W. GODWARD.)—Let O_1, O_2, O_3 be the centres of the escribed circles having internal contact with the sides BC, CA, AB respectively of the triangle ABC, and P any other point in the same plane; then (1) if R_1, R_2, R_3 be the intersections of AP, BP, CP with BC, CA, AB, respectively, the triad (O_1R_1, O_2R_2, O_3R_3) will be concurrent; and (2) if S_1, S_2, S_3 be the intersections of O_1P, O_2P, O_3P with BC, CA, AB, respectively, the triad (AS_1, BS_2, CS_3) will be concurrent.

Solution by S. BILLS; W. H. LAVERTY; J. DALE; *and the* PROPOSER.

The first part of the question will, evidently, only be a particular case of the following much more general and comprehensive theorem. Let ABC be any plane triangle, and from its angular points let lines be drawn through *any* assumed point, meeting the opposite sides in A_1, B_1, C_1; describe the triangle $A_1B_1C_1$, and from its angular points draw lines through *any other* assumed point, meeting the opposite sides in A_2, B_2, C_2; then will the lines AA_2, BB_2, CC_2 be concurrent. For, from the triangles AA_2B_1, AA_2C_1; &c., &c.; we readily obtain



$$\frac{\sin A_2AC_1}{\sin A_2AB_1} = \frac{A_2C_1 \cdot AB_1}{A_2B_1 \cdot AC_1}, \quad \frac{\sin B_2BA_1}{\sin B_2BC_1} = \frac{B_2A_1 \cdot BC_1}{B_2C_1 \cdot BA_1},$$

$$\frac{\sin C_2CB_1}{\sin C_2CA_1} = \frac{C_2B_1 \cdot CA_1}{C_2A_1 \cdot CB_1};$$

hence, multiplying together these three equations, and remembering that $A_2C_1 \cdot B_2A_1 \cdot C_2B_1 = A_2B_1 \cdot B_2C_1 \cdot C_2A_1$, $AB_1 \cdot BC_1 \cdot CA_1 = AC_1 \cdot BA_1 \cdot CB_1$, the criterion of the concurrence of AA_1, BB_1, CC_1 is satisfied, *viz.*,

$$\sin A_2AC_1 \cdot \sin B_2BA_1 \cdot \sin C_2CB_1 = \sin A_2AB_1 \cdot \sin B_2BC_1 \cdot \sin C_2CA_1.$$

[Mr. BILLS gives at length an independent proof of the property (2), with another diagram. But from what is given above, we may, in fact, prove the following theorem, which includes, as particular cases, *both* those in the Question, together with that in Question 1616 (*Reprint*, Vol. III., p. 29).

If of three triangles, each inscribed in the preceding, any two are in homology with the third, they are in homology with each other.

For let the figure represent any three such triangles; then, putting

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = p, \quad \frac{A_2B_1}{A_2C_1} \cdot \frac{B_2C_1}{B_2A_1} \cdot \frac{C_2A_1}{C_2B_1} = q,$$

$$\frac{\sin A_2AC_1}{\sin A_2AB_1} \cdot \frac{\sin B_2BA_1}{\sin B_2BC_1} \cdot \frac{\sin C_2CB_1}{\sin C_2CA_1} = r,$$

and introducing appropriate *signs*, Mr. BILLS' equations give $pqr = -1$. If therefore any two of the three quantities (p, q, r) are equal to -1, so likewise is the third; that is to say, (by Ceva's theorem) if any two of the three triangles (whose criterions of homology, in pairs, are $p = -1$, $q = -1$, $r = -1$) are homologous (or co-polar) with the third, they are homologous with each other.

After the above was in type we observed that the theorem is proved, but in a different manner (by the method of anharmonic ratio), in Townsend's *Modern Geometry*, Vol. II., p. 144.]

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Solutions of Questions 1713, 1716, 1734; and Demonstrations of some of the Properties of the Regular Tricusp enunciated in Steiner's Paper (*Reprint* Vol., III., pp. 97—100). By J. Dale.

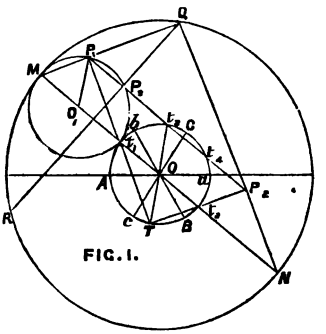

FIG. I.

1. Let O (Fig. 1) be the common centre of the circles ABC, MQN, the radius of the latter being three times that of the former; draw any radius Ot_1M making an angle θ with the radius OA; take $OO_1 = \frac{2}{3}OM$; from O_1 draw $O_1P_1 = OM$ and making an angle 3θ with OO_1: then it is evident that P_1 is a point on the hypocycloid of three branches (or *tricusp*) touching the interior circle in A, B, C, the points of trisection of the circumference, and having Aa, Bb, Cc for cuspidal tangents. Join P_1M, P_1t_1, and produce P_1t_1 to cut ABC in T; then from the nature of roulettes, P_1M and P_1t_1T are respectively the normal and tangent to the tricusp at P_1. Produce MO, cutting the circles in t_2, N, and join TO; then the triangles $P_1O_1t_1$, TOt_1, are equal in all respects; hence the arc $At_1 = \frac{1}{3}AT$, and the chord $P_1t_1 = Tt_1$; and since the arc AB = BC = CA, we have $Ct_1 = \frac{1}{3}CT$, and $BCt_1 = \frac{1}{3}BACT$. Therefore (I) *the arcs into which the inscribed circle is divided by the tangent are divided in the ratio of* 1 : 2 *by the cuspidal tangents; and* (II) *the chord intercepted by the inscribed circle on any tangent is bisected at the point nearest the point of contact of the tricusp and circle; and conversely, any line satisfying these conditions is a tangent to the tricusp.*

2. Join Tt_2, then the arc $AB = 2aB$, and $AT = 2At_1 = 2at_2$; hence $Bt_2 = \frac{1}{2}BT$, and therefore, by (I), Tt_2 is a tangent to the tricusp; and making $t_2P_2 = t_2T$, P_2 is, by (II), the point of contact. Join P_1P_2, cutting the inscribed circle in t_3, t_4; then as P_1P_2 is parallel to t_1t_2 and $= 2t_1t_2$, t_3 lies on TO, and the arc $t_1t_3 = t_2t_4 = Tt_2 = \frac{1}{2}Tt_4$; therefore $At_3 = \frac{1}{2}At_4$, also $At_3C = \frac{1}{2}At_4C$, therefore $Ct_3 = \frac{1}{2}Ct_4$; hence P_1P_2 is a tangent, and if $t_3P_3 = t_3t_4$, P_3 is the point of contact. It may also be readily shown from the figure that the normals at P_1, P_2, P_3 meet in a point Q on the circumscribed circle. Therefore, (III) *tangents at right angles intersect on the inscribed circle;* (IV) *the chord of contact* P_1P_2 *is of constant length* $(4r)$, *and* (V) *is also a tangent to the curve;* (VI) *the normals at any three points, the tangents at which form a right-angled triangle, meet in the circumscribed circle;* (VII) *the chord intercepted by the circumscribed circle on any normal is divided in the ratio of* 1 : 2 *at the point of contact.*

3. QM is parallel to Tt_2, and if the cuspidal tangent Bb be produced to meet the circumscribed circle, it will divide the arc QM in the ratio of 1 : 2; therefore (VIII) *the normal to the tricusp at any point is a tangent to another tricusp touching the circumscribed circle at the cusps of the former; and conversely, the tangent to the tricusp at any point is a normal to another tricusp having the inscribed circle of the former for its circumscribed circle, the cusps of the one corresponding to the points of contact of the other.* In other words, *the evolute and involute of the tricusp are similar tricusps.*

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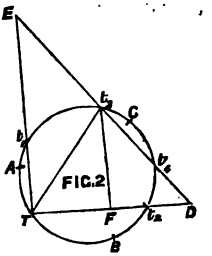
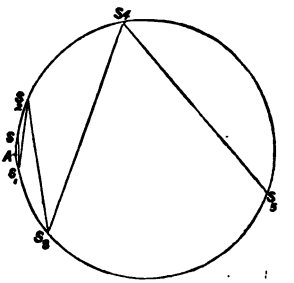
4. Let ABC (Fig. 2) be the inscribed circle of a tricusp; A, B, C being the points of contact; Tt_1, Tt_2 two tangents at right angles; through any point t_3 on the circle draw a straight line so that the portion DE intercepted between Tt_1, Tt_2 may be bisected in t_3. Join Tt_3; then the arc $t_1t_3 = \frac{1}{2}Tt_2t_4$, and as $At_1 = \frac{1}{2}AT$, we have $At_1t_3 = \frac{1}{2}At_2t_4$; therefore, by (I), DE is a tangent to the tricusp. Draw t_3F parallel to ET, then the angle Tt_3D is bisected by t_3F. Therefore, (IX) *if the intercept of a line between two rectangular tangents is bisected by a point in the inscribed circle, it is enveloped by the tricusp*; and, (X) *if the vertex of a variable angle moves on the inscribed circle, and one of the sides turns round a fixed point on the circle, while the bisectors are constantly parallel to a fixed pair of rectangular tangents intersecting at the point; then the other side of the variable angle is enveloped by the tricusp.*



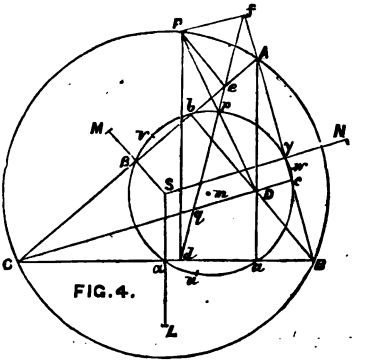
5. Let ss_1 (Fig. 3) be any arc of a circle, and let $sA = \frac{1}{2}s_1A$, then by (I) the chord ss_1 is a tangent to the circumscribed tricusp; and drawing s_1s_2 perpendicular to the diameter through s, or what is the same thing, making the arc $s_1s_2 = 2ss_1$, then the arc $As_2 = 2As_1$, therefore s_1s_2 is a tangent; and drawing according to the same law the chords s_2s_3, s_3s_4, s_4s_5, &c., these chords are also tangents. To find the distance of any one of these points from s, let $ss_1 = a$; then we have $ss_1 = a$, $ss_2 = -a$; $ss_3 = (1+2)a$,



$$ss_4 = -(1+4)a\,;\ ss_5 = (1+2+8)a,\ ss_6 = -(1+4+16)a\,;$$
$$ss_7 = (1+2+8+32)a,\quad ss_8 = -(1+4+16+64)a,$$
$$ss_n\ (n \text{ odd}) = \left\{1+2\,(1+2^2+2^4.... +2^{n-3})\right\}a = \tfrac{1}{3}\,(1+2^n)\,a,$$
$$ss_n\ (n \text{ even}) = -\,(1+2^2+2^4.... +2^{n-2})\,a = \tfrac{1}{3}(1-2^n)\,a\,;$$

and both these forms are included in $ss_n = \frac{1}{3}\left\{1-(-2)^n\right\}a$.

6. Let u, v, w (Fig. 4) be the points of contact of the tricusp; m the centre of the inscribed circle; draw two pairs of rectangular tangents, having their vertices in b and c respectively, and determining by their intersections the quadrilateral ABCD; then, by (IX), AC and AB are bisected in β and γ, the points where they are cut by the circle (m); therefore (m) is the nine-point circle of the triangle ABC, and consequently passes through the middle point of BC; therefore, by (IX), BC is a tangent to the tricusp; and as AD meets BC on the circle (m) at right angles, AD also is a tangent. Also $AD^2 + BC^2 = Ac^2 + Dc^2 + Cc^2 + Bc^2 = AC^2 + BD^2$



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$= Ca^2 + Aa^2 + Ba^2 + Da^2 = AB^2 + CD^2 = 16r^2(\sin^2 C + \cos^2 C) = 16r^2$. If L, M, N, the centres of the circles drawn round BCD, CDA, ADB, be joined, the triangle LMN, as is well known, will be equal and similar to ABC, the corresponding sides BC, MN; CA, NL; AB, LM will be parallel, and the circle (m) will be the common nine-point circle; therefore the sides MN, NL, LM, and the perpendiculars LS, MS, NS, touch a tricusp, having (m) for its inscribed circle, and its points of contact opposite to u, v, w.

7. Let d, e, f be the feet of perpendiculars drawn from any point P in the circumscribed circle on the sides of the triangle ABC; join DP; then def and DP intersect in a point p on the nine-point circle and $Pp = Dp$ (see Quest. 1649, *Reprint*, Vol. III., p. 58), hence the triangles pfP, pqD are equal in every respect, whence $pf = pq$; therefore, since the straight line fq intercepted between the rectangular tangents, cC, cA is bisected in p by the inscribed circle (m), it follows by (IX) that def is a tangent to the tricusp having the circle (m) for its inscribed circle.

1551. (Proposed by the Rev. J. BLISSARD.)—

Using the notation of Spence's Transcendents, viz.,

$$L^n(1+x) = \frac{x}{1^n} - \frac{x^2}{2^n} + \frac{x^3}{3^n} - \frac{x^4}{4^n} + \text{\&c., prove that}$$

$$(1)\ldots. L^2(1+x) + L^2\left(\frac{1}{1+x}\right) = \tfrac{1}{2}\log^2(1+x);$$

$$(2)\ldots. L^2\left(\frac{1}{1+x}\right) + L^2\left(\frac{x}{1+x}\right) = \log^2(1+x) - \log x \log(1+x) - \tfrac{1}{6}\pi^2.$$

Solution by the PROPOSER.

It has been shown on p. 59 of Vol. II. of the *Reprint*, that, if B is the Representative of Bernoulli's numbers, we shall have

$(1+x)^B = x^{-1}\log(1+x) = 1 - \frac{1}{2}x + \frac{1}{6}x^2 - \text{\&c.}$; hence, integrating,

$$\frac{(1+x)^{B+1}-1}{B+1} = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \text{\&c.} = L^2(1+x);$$

also putting $\frac{-x}{1+x}$ for x, we have $\frac{(1+x)^{-(B+1)}-1}{B+1} = L^2\left(\frac{1}{1+x}\right)$;

therefore $L^2(1+x) + L^2\left(\frac{1}{1+x}\right) = 2\left\{(B+1)\frac{\log^2(1+x)}{1.2} + \text{\&c.}\right\}$.

But $(B+1)^n = 0$, (n odd and >1); also $B+1 = B_1 + 1 = \frac{1}{2}$; hence we obtain the formula (1).

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Again, since $\frac{1}{x} = \frac{1}{1+x} - \frac{1}{1-(1+x)^{-1}} = \frac{1}{1+x} + \frac{1}{(1+x)^2} + \&c.$,

$$L^2(1+x) = \int \frac{1}{x} \log(1+x)\, dx = \int \log(1+x)\, dx \left\{ \frac{1}{1+x} + \frac{1}{(1+x)^2} + \&c. \right\}$$

$$= \tfrac{1}{2}\log^2(1+x) - \left\{ \frac{1}{1^2} \cdot \frac{1}{1+x} + \frac{1}{2^2} \cdot \frac{1}{(1+x)^2} + \&c. \right\}$$

$$- \log(1+x) \left\{ \frac{1}{1+x} + \frac{1}{2} \cdot \frac{1}{(1+x)^2} + \&c. \right\}$$

$$= -\tfrac{1}{2}\log^2(1+x) + L^2\left(\frac{x}{1+x}\right) + \log(1+x)\log x + C.$$

To determine C, let $x = 1$; then $L^2(2) = -\frac{1}{2}\log^2(2) + L^2(\frac{1}{2}) + C$; but from (1), putting $x = 1$, $L^2(2) + L^2(\frac{1}{2}) = \frac{1}{2}\log^2(2)$;

therefore $C = 2L^2(2) = 2\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \&c.\right) = \frac{\pi^2}{6}$; hence

$$L^2(1+x) - L^2\left(\frac{x}{1+x}\right) = \frac{\pi^2}{6} + \log x \log(1+x) - \tfrac{1}{2}\log^2(1+x) \ldots (1').$$

Taking (1') from (1), we obtain the formula (2).

COROLLARY.—It is easy to show that

$$\log^2(1+x) = 2\left(\Sigma_1 \cdot \frac{x^2}{2} - \Sigma_2 \cdot \frac{x^3}{3} + \Sigma_3 \cdot \frac{x^4}{4} - \&c.\right),$$

where $\Sigma_n = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}$; hence from (1), by equating coefficients, we obtain the following known property of numbers; viz.,

$$\Sigma_n = \frac{n}{1} - \frac{n(n-1)}{1.2^2} + \frac{n(n-1)(n-2)}{1.2.3^2} - \&c.$$

1784. (Proposed by T. T. WILKINSON, F.R.A.S.)—If from any given point in an ellipse there be drawn an ordinate to either axis, and if from the point where this meets the curve again a diameter be drawn, and to this diameter another ordinate be applied from the given point; then the point in which this ordinate meets the ellipse shall be in the circumference of the circle of curvature at the given point.

Solution by R. WARREN, B.A.; M. COLLINS, B.A.; *and others.*

Let C be the centre of the conic, and Q the point in which the circle of curvature at P meets it again. It is a well known theorem, that if a circle intersect a conic, its chords of intersection make equal angles with the axis. In the case of the circle of curvature, the tangent at P is one chord of inter-

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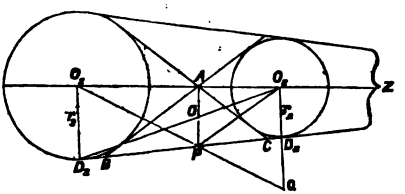
section and the line PQ the other; we have, therefore, only to draw PQ, making the same angle with the axis at the tangent at P, and we have the point Q. Now since the tangent at P is an ordinate to the diameter CP, the line PQ is by symmetry an ordinate to the reflexion of CP (with respect to the axis), which is nothing more than the diameter used in the construction given in the Question.

1697. (Proposed by MATTHEW COLLINS, B.A.)—If two circles touch a straight line a, and have a different kind of contact with two other straight lines b and c, prove that the middle point of the distance of a from the point of intersection of b and c lies in a straight line with the centre of one of the two circles and the point where the other circle touches a.

N.B.—The contact of two circles with two straight lines is said to be like, or of the same kind, when the two circles are on the *same side* of *both* lines, or else on *different sides* of *both* lines; but if the two circles lie on the same side of one of the two straight lines, and on different sides of the other straight line, then the contact of the two circles with the two straight lines is said to be unlike, or of different kinds.

Solution by J. DALE; E. FITZGERALD; H. MURPHY; *and others.*



This Question may also be enunciated as follows. If O_2, O_3 be the centres of the escribed circles, touching the sides CA, AB, respectively, of the triangle ABC; and if D_2, D_3 be the points where these circles touch BC; then the straight lines O_2D_3, O_3D_2 intersect in the middle point O of the perpendicular AP from A on BC. Let O_2D_3 meet AP in O; then $O_2D_2 : D_3D_3 = OP : PD_2$. But

$$O_2D_2 = r_2 = \frac{\Delta}{s_2},\quad D_2D_3 = s + s_1 = b + c,\quad \text{and}\quad PD_2 = s - b\cos C = \frac{(b+c)s_2}{a};$$

hence the above proportion gives $OP = \frac{\Delta}{a} = \frac{1}{2}AP$.

In like manner it may be proved that O_3D_2 also passes through O.

1698. (Proposed by MATTHEW COLLINS, B.A.)—Prove that the foot of a perpendicular on a common tangent (a) to two circles, drawn from the point of intersection of the two contrary common tangents (b and c) to those circles, lies in a straight line with the centre of one of the circles and the reflexion of the centre of the other with respect to the tangent (a).

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Solution (1) *by* ABRACADABRA; (2) *by* G. O. HANLON; *and others.*

[See Figure to preceding Solution (Quest. 1697).]

1. Since $\frac{r_3}{O_3Z} = \frac{AP}{AZ} = \frac{r_2}{O_2Z}$, $\therefore \frac{r_3 - AP}{AP - r_2} = \frac{AO_3}{AO_2} = \frac{r_3}{r_2}$, whence $AP = \frac{2r_2r_3}{r_2 + r_3}$. [This, we may observe, proves that the perpendicular from an angle on the opposite side of a triangle is an harmonic mean between the radii of the circles escribed to the other two sides, a property used in the solution of Quest. 1663.] Now $\frac{r_3}{r_2 + r_3} = \frac{AO_3}{O_2O_3}$, and, if Q be the reflexion of O_2 with respect to D_3D_3, we have $2r_2 = O_2Q$; hence it follows that $AP : AO_3 = O_2Q : O_2O_3$; and therefore Q, P, O_3 are in a straight line.

2. *Otherwise:* Since $O_3A : AO_2 = O_3Z : ZO_2$, and P lies in the semicircle on AZ, therefore $O_3P : PO_2 = O_3A : AO_2$, hence $\angle O_3PA = APO_2$, and therefore $\angle O_3PD_3 = O_2PD_2 = QPD_2$; whence it follows that Q, P, O_3 are a straight line.

1756. (Proposed by W. S. BURNSIDE, B.A.)—If normals drawn to a central quadric at the points (1), (2), (3), (4), (5), (6) pass through a point, and r_{12} denote the length of the semi-diameter parallel to the chord (1, 2); prove that

$$r_{12}^2 + r_{34}^2 + r_{56}^2 = a^2 + b^2 + c^2,$$

where a, b, c are the semi-axes of the quadric.

Solution by the PROPOSER.

1. To construct the normals to a quadric which lie in a plane, draw a perpendicular from the pole of the plane on it, and draw also two tangent planes to the surface through this line; the points of contact of these planes are the points at which the normals should be drawn. This being so, if n_1 and n_2 be the lengths of two intersecting normals at the points (1, 2) and p_1, p_2 the central perpendiculars on the tangent planes at the same points, then $n_1p_1 + n_2p_2 = 2r_{12}^2$. For if S be the result of substituting the coordinates of the middle point of the chord in the equation of the surface; ϖ_1, ϖ_2 the perpendiculars from that point on the tangent planes; and $2C_{12}$ the length of the chord; then, plainly, $n_1\varpi_1 + n_2\varpi_2 = 2C_{12}^2$; also $C_{12}^2 = r_{12}^2 S$, $\varpi_1 = p_1 S$, $\varpi_2 = p_2 S$; hence $n_1p_1 + n_2p_2 = 2r_{12}^2$. (See Salmon's *Conics*, 4th ed., p. 227.)

2. If (x, y, z) be the coordinates of the point through which all the normals pass, we have, from the equation of the normal at (1),

$$\frac{x_1 - x}{a^{-2}x_1} = \frac{y_1 - y}{b^{-2}y_1} = \frac{z_1 - z}{c^{-2}z_1} = n_1p_1 = \lambda_1 \text{ (say)},$$

$\left(\text{as } n_1^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2, \text{ and } p_1^{-2} = a^{-4}x_1^2 + b^{-4}y_1^2 + c^{-4}z_1^2\right)$;

therefore $x = x_1\left(1 - \frac{\lambda_1}{a^2}\right)$, $y = y_1\left(1 - \frac{\lambda_1}{b^2}\right)$, $z = z_1\left(1 - \frac{\lambda_1}{c^2}\right)$;

but $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$, $\therefore \frac{a^2x^2}{(a^2 - \lambda)^2} + \frac{b^2y^2}{(b^2 - \lambda)^2} + \frac{c^2z^2}{(c^2 - \lambda)^2} = 1$,

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or $\lambda^6 - 2\lambda^5(a^2+b^2+c^2) + \lambda^4 L + \lambda^3 M + \lambda^2 N + \lambda R + S = 0$, where

$$R = 2a^2b^2c^2\left\{(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b)z^2 - b^2c^2 - c^2a^2 - a^2b^2\right\}$$

$$S = a^4b^4c^4\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right);\ \&c.\ \&c.;$$

whence $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 2(a^2+b^2+c^2)$.

But $\lambda_1 + \lambda_2 = 2r_{12}^2 = (n_1p_1 + n_2p_2)$, by (1);

we have then finally $r_{12}^2 + r_{34}^2 + r_{56}^2 = a^2+b^2+c^2$.

3. From the last two coefficients of the above equation of the sixth degree it is easily seen that, if (a_1, a_2, a_3) are the elliptic coordinates of the point (x, y, z) through which the normals pass, we have

$$\frac{1}{2}\sum_{r=6}^{r=} \frac{1}{n_rp_r} = \frac{1}{a^2-a_1^2} + \frac{1}{a^2-a_2^2} + \frac{1}{a^2-a_3^2}.$$

1743. (Proposed by the EDITOR.)—1. If P, Q be two points in the path of a projectile; PT, QT the tangents at P, Q; and α, β, γ the angles made with the horizon by PT, TQ, QP, respectively; prove geometrically that $\tan\alpha + \tan\beta = 2\tan\gamma$.

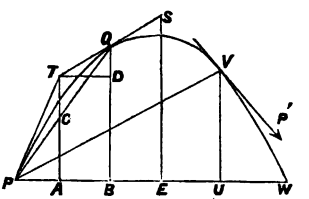
2. Hence show that if a particle be projected from one end of the horizontal base of an isosceles triangle so as exactly to shoot down the opposite side; and if α be the angle of projection, γ the angle at the base of the triangle; then $\tan\alpha = 3\tan\gamma$.

3. Prove that the velocities at P, Q are proportional to PT, QT.

Solution by J. TAYLOR; E. MCCORMICK; J. DALE; *and others.*

1. Through T, Q draw the diameters TA, QB to the horizontal line through P; also draw TD parallel to PB; and let TA meet PQ in C. Then, since the diameter through the intersection of two tangents to a parabola bisects the chord of contact, we have PC = CQ, and therefore PA = AB = TD; hence

$$\tan\alpha + \tan\beta = \frac{TA}{AP} + \frac{QD}{DT} = 2\frac{QB}{BP} = 2\tan\gamma.$$



2. Here PV, one side of the isosceles triangle PVP', is a chord, and the other side VP' is a tangent, to the parabolic path PVW; moreover $\tan\beta = -\tan\gamma$; therefore, by (1), we have $\tan\alpha = 3\tan\gamma$.

3. In PB produced make BE = PA = AB, and produce TQ to meet the diameter through E in S; then TQ = QS; hence, since the horizontal velocity of a projectile is uniform, the velocities at P, Q are proportional to PT, QS, that is to PT, QT.

[From (3) we obtain a simple proof of Professor SYLVESTER'S theorem (Quest. 1439; *Reprint*, Vol. II., p. 55), since the velocities at P, Q are known to be as the distances of these points from the directrix, or from the focus.]

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1229. (Proposed by S. WATSON.)—Show that the average area of all the triangles that can be formed by joining three points, taken *ad libitum* upon the surface of a given triangle, is one-twelfth of the triangle.

Solution by PROFESSOR SYLVESTER.

Let m be the average value of any quantity of the i^{th} order of linear magnitude depending solely on the internal constitution of a group of n elements taken arbitrarily in a triangle; let m be the average value of such quantity when three of the elements are limited respectively to the three sides, and μ_1, μ_2, μ_3, its values when one of them is limited to an angle and another to an opposite side; it may be proved that

$$\mathrm{M} = \frac{8n(n-1)(n-2)}{(2n+i)(2n+i-1)(2n+i-2)} m + \Sigma \frac{4n(n-1)}{(2n+i)(2n+i-1)(2n+i-2)} \mu \quad \ldots (\Omega),$$

provided $2n+i-2$ is greater than zero. In the case before us, $n=3$, $i=2$, and consequently $\mathrm{M} = \frac{1}{12}(2m+\mu_1+\mu_2+\mu_3)$. Call A the area of the triangle; then it is easily seen that $m = \frac{1}{4}\mathrm{A}$; m, in fact, being the area of the triangle obtained by joining together the middle points of the sides of the given triangle. Again, as regards μ_1, corresponding to one point at the angle A, a second at the side BC, and a third roving anywhere over the surface of the triangle, it is easily seen that, taking any point P in BC, and putting BC $= a$, BP $= x$, we have

$$a \,.\, \mathrm{A} \,.\, \mu_1 = \frac{\mathrm{A}^2}{3} \int_0^a dx \, \frac{x^2 + (a-x)^2}{a^2} = \frac{2a\mathrm{A}^2}{9};$$

hence $\mu_1 = \frac{8}{9}\mathrm{A} = \mu_2 = \mu_3$; and therefore $\mathrm{M} = \frac{1}{12}(\frac{2}{4} + \frac{8}{9})\mathrm{A} = \frac{1}{16}\mathrm{A}$.

NOTE.—This solution is offered in exemplification of the power of the general theorem marked (Ω). An analogous theorem, giving rise to a depression of *four* degrees in the integrations, applies to the tetrahedron, and in general the summations of any intrinsic affection of a group of elements, whether free or conditional, for any plane polygon may be reduced *three* and for any polyhedron *four* degrees; the condition or conditions to which the group is subject being understood to be intrinsic conditions of form, and not involving linear magnitude. Probability questions come under the head of conditional groups, and the value of i, the degree of the dimensions of the affection, is for such questions *zero*. By an *average* value of an intrinsic affection of a conditional group of elements, in the application of this method, is to be understood the aggregate of the affection divided by a suitable power of the range, or more generally by the product of the ranges of the several elements. The *mean* value in its ordinary sense will be the average value thus defined of the intrinsic affection of the conditional group *divided* by the *probability* of the fulfilment of the conditions. Any such conditions may either be algebraical; as for instance that the triangle cornered by three elements shall be obtuse, or that the ratio of no two sides shall exceed a given quantity; or it may be transcendental, as, for instance, that the quadrilateral cornered by four elements shall be convex. Provided only that each such condition be intrinsic and independent of the scale of linear magnitude, the theorems of reduction apply universally, being founded on primordial conceptions of simple and compound shnilitude.

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1739. (Proposed by H. McColl.)—Let $f(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$. Let $f_1(x) = A_{n-1}x^{n-1} + 2A_{n-2}x^{n-2} + \ldots + (n-1)A_1 + nA_0$. Form $f_2(x)$ from $f_1(x)$ in the same manner, and so on to $f_n(x)$. Let a and b be any two quantities *of the same sign*, of which a is greater numerically than b. Show that if the series of functions $f(x)$, $f_1(x)$, $f_2(x) \ldots f_n(x)$ give p changes of signs when $x = a$, and q changes when $x = b$, the number of real roots of $f(x) = 0$ between a and b cannot exceed $p - q$.

Solution by the Proposer.

Let $y = x^{-1}$, and let $\phi(y) = 0$ be the equation whose roots are the reciprocals of the roots of $f(x) = 0$. Then $f(x) = x^n \phi(y)$; and generally denoting the rth derived function of $\phi(y)$ by $\phi^r(y)$ we shall have $f_r(x) = x^{n-r}\phi^r(y)$; so that when x is positive, $f_r(x)$ and $\phi^r(y)$ have the same sign; and when x is negative, $f_r(x)$ and $\phi^r(y)$ have the same or opposite signs according as $n - r$ is even or odd. If therefore the series of functions $f(x)$, $f_1(x)$, $f_2(x)$, $\ldots f_n(x)$ give p changes of signs when $x =$ a positive quantity a, and q changes when $x =$ a positive quantity b, the series of functions $\phi(y)$, $\phi'(y)$, $\phi^2(y)$, $\ldots \phi^n(y)$ will also give p changes when $y = a^{-1}$, and q changes when $y = b^{-1}$. But in this case, by Fourier's theorem, the number of real roots of $\phi(y) = 0$ between a^{-1} and b^{-1} (which is the same as the number of real roots of $f(x) = 0$ between a and b) cannot exceed $p - q$. Again, if the series of functions $f(x)$, $f_1(x)$, $f_2(x), \ldots f_n(x)$ give p changes of signs when $x =$ a negative quantity a, and q changes when $x =$ a negative quantity b, the series of functions $\phi(y)$, $\phi'(y)$, $\phi^2(y)$, $\ldots \phi^n(y)$ will give $n - p$ changes when $y = a^{-1}$ and $n - q$ changes when $y = b^{-1}$. But in this case also, by Fourier's theorem, the number of real roots of $\phi(y) = 0$ between a^{-1} and b^{-1} (that is, the number of real roots of $f(x) = 0$ between a and b) cannot exceed $(n - q) - (n - p)$, that is, $p - q$.

N.B.—If a and b be taken sufficiently near, and the equation $f(x) = 0$ have no equal roots, this theorem will always inform us of the exact number of real roots between a and b when Fourier's theorem leaves the matter doubtful. From this, and the similarity between the two theorems, the one may be called the complement of the other.

Since $f_1(x)$ evidently $= nf(x) - xf'(x)$, whatever value we give to x, and $f'(x) = x^{-1}\{nf(x) - f_1(x)\}$, it is evident that if $f'(a)$ and $f(a)$ have opposite signs (a being positive), $f_1(a)$ and $f(a)$ will have the same sign; and, if $f_1(a)$ and $f(a)$ have opposite signs, $f'(a)$ and $f(a)$ will have the same sign. Attention to these facts will often prevent unnecessary labour in ascertaining the sign of $f'(a)$ or $f_1(a)$, when we employ this theorem and Fourier's simultaneously.

1761. (Proposed by H. McColl.)—Let $f(x) = 0$ be any equation of the nth degree. Let $f^r(x)$ be the rth derived function of $f(x)$. Let it be

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known that $f^r(x) = 0$ has only one root between the two quantities a and b. Show that by giving a more general expression to the trial fraction in Horner's method of approximation, we can simultaneously approximate as nearly as we please to the root of $f^r(x) = 0$, separate any two roots of $f^{r-1}(x) = 0$ that may exist between a and b, and obtain information as to the number of roots of $f^{r-2}(x) = 0$, $f^{r-3}(x) = 0 \ldots . f'(x) = 0$, $f(x) = 0$ passed over in the operation. Show also that the same process will enable us to approximate simultaneously to any maximum or minimum value of $f(x)$, and to the value of x, which makes $f(x)$ a maximum or minimum.

Solution by the PROPOSER.

1. Let x_0 denote the root of $f^r(x) = 0$ between a and b. Let a_0 be any approximation to x_0. Assuming $x_0 = a_0 + x_1$, we get $f^r(a_0 + x_1) = f^r(a_0) + f^{r+1}(a_0)\,x_1 +$ terms involving higher powers of x_1. If a_0 is a sufficiently near approximation to x_0, these terms are numerically small compared with the first two, and $x_1 = -\dfrac{f^r(a_0)}{f^{r+1}(a_0)}$ nearly.

2. Again; $f(x_0) = f(a_0 + x_1) = f_1(x_1)$ say $= A_n x_1^n + A_{n-1} x_1^{n-1} + \ldots + A_1 x_1 + A_0$ say. The coefficients A_0, A_1, A_2, &c., may be found successively by Horner's method; and we know from Taylor's theorem that $f^r(a_0) = \lfloor r\, A_r$, and $f^{r+1}(a_0) = \underline{\lfloor r+1}\, A_{r+1}$.

But it has been shown that $x_1 = -\dfrac{f^r(a_0)}{f^{r+1}(a_0)}$ nearly;

therefore $x_1 = -\dfrac{\lfloor r\, A_r}{\underline{\lfloor r+1}\, A_{r+1}}$ nearly; that is, $-\dfrac{A_r}{(r+1)\, A_{r+1}}$ nearly.

3. Again; assuming $a_1 = a_1 + x_2$, (a_1 being any approximation to x_1,) we get $f(x_0) = f(a_0 + a_1 + x_2) = f_1(a_1 + x_2) = f_2(x_2)$ say $= A_n x_2^n + A_{n-1} x_2^{n-2} + \ldots + A_1 x_2 + A_0$ say. The new values of the coefficients A_0, A_1, A_2, &c., may again be found successively by Horner's method from their former values in $f_1(x_1)$, and by the same reasoning as before we have

$$x_2 = -\frac{A_r}{(r+1)\, A_{r+1}} \text{ nearly.}$$

4. If we continue this process and assume x_0 successively $= a_0 + x_1 = a_0 + a_1 + x_2 = a_0 + a_1 + a_2 + x_3 =$ &c., a_m being any approximation to x_m, representing the expansion of $f_{m-1}(a_{m-1} + x_m)$ by $f_m(x_m)$, and denoting generally the coefficient of x_m^r by A_r, we shall get $-\dfrac{A_r}{(r+1)\, A_{r+1}}$ as an approximation to x_m which gets closer and closer as m increases.

5. The approximation a_m to which we are guided by $-\dfrac{A_r}{(r+1)\, A_{r+1}}$ as a trial fraction is perfectly arbitrary, and may be taken numerically

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greater or less than x_m, but the latter more conveniently, as then $a_0, a_1, a_2, \ldots a_m$ will all have the same sign. The approximation a_m is numerically greater or less than x_m according as A_r changes or retains its sign in passing from its value in $f_m(x_m)$ to its value in $f_{m+1}(x_{m+1})$.

6. The limit which $f_m(x_m)$ approaches as m increases is

$$f(x_0) + f'(x_0)x_m + \frac{f^2(x_0)}{\lfloor 2} x_m^2 + \ldots + \frac{f^n(x_0)}{\lfloor n} x_m^n, \text{ in which } \frac{f^r(x_0)}{\lfloor r},$$

the limit to which A_r approaches, is necessarily zero, since x_0 is by hypothesis the root of $f^r(x) = 0$.

7. If the equation $f^{r-1}(x) = 0$ has two real and nearly equal roots between a and b, the root x_0 of $f^r(x) = 0$ is situated between them: so that our approximation to x_0 is also an approximation to each of the roots of $f^{r-1}(x) = 0$ till one of these roots is passed over, which will be indicated by A_{r-1} changing its sign.

8. When A_{r-1} retains its sign as it approaches and when it reaches its limit $\frac{f^{r-1}(x_0)}{\lfloor r-1}$, the equation $f^{r-1}(x) = 0$ cannot have two real roots between a and b; and we may be sure that this is the case as soon as we find that $rA_r b_m + A_{r-1}$ has the same sign as A_{r-1}. b_m being a numerically superior approximation to x_m obtained by the trial fraction.

9. When $f^{r-1}(x) = 0$ has two equal roots between a and b, the limit to which A_{r-1} approaches, as well as the limit to which A_r approaches, is zero.

10. And generally if $f^{r-s}(x) = 0$ has $s+1$ real and nearly equal roots between a and b, the root x_0 of $f^r(x) = 0$ is situated between the greatest and least; so that our approximation to x_0 is also an approximation to each of the $s+1$ roots of $f^{r-s}(x) = 0$ till p roots are passed over, which will be indicated by a decrease or increase (according as a_m was positive or negative) of p signs in those given by the series of coefficients $A_{r+1}, A_r, A_{r-1}, \ldots A_{r-s}$.

11. If these $s+1$ roots are equal, all the coefficients $A_r, A_{r-1}, A_{r-2}, \ldots A_{-s}$ approach the limit zero.

12. If when we know that no more than $s+1$ real roots of $f^{r-s}(x) = 0$ exist between a and b, the coefficients $A_r, A_{r-1}, A_{r-2}, \ldots A_{r-(s-1)}$ approach the limit zero, while A_{r-s} without changing its sign approaches a limit which is *not* zero; then $f^{r-s}(x) = 0$ cannot have two real roots between a and b; and if we find that $f^{r-s}(x) = 0$ has no real root between a and b, then $f^{r-s-1}(x) = 0$ cannot have two real roots in the same interval.

13. When from the rapid numerical decrease of $A_r, A_{r-1}, A_{r-2}, \ldots A_1, A_0$ we suspect that $f(x) = 0$ has $r+1$ equal roots each $= x_0$, the follow-

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ing tests will immediately remove all uncertainty on the point, provided that A_n, A_{n-1}, ... A_1, A_0, the coefficients in $f(x)$, are integers.

First, when there is only one doubtful interval. Let c be any root of $f(x) = 0$ repeated q times. Then $A_n c$ is a whole number, and $(A_n x - A_n c)\, M^{-1}$, in which M is the greatest common measure of A_n and $A_n c$, will divide $f(x)$ q times.

Next, when there are two doubtful intervals. If c and c_1 be two roots of $f(x) = 0$, each of which is repeated q times; then $A_n (x-c)(x-c_1)\, M^{-1}$, that is $\{A_n x^2 - A_n (c+c_1) x + A_n c c_1\}\, M^{-1}$, in which M is the greatest common measure of the *integral* coefficients A_n, $A_n (c+c_1)$, $A_n c c_1$, will divide $f(x)$ q times.

The same principle may evidently be extended to any number of doubtful intervals.

14. When $f(x)$ has a maximum or minimum value; since x_0 is a real root of $f''(x) = 0$, and the limit to which A_0 approaches as m increases is $f(x_0)$, by Art. 6, it is evident that $f(x_0)$ will, if we give the proper value to r, be a maximum or minimum value of $f(x)$.

On the Numerical Solution of Equations.

(Continued from the Solutions of Questions 1739, 1761.)

Form Fourier's functions and the functions given in Quest. 1739, and apply both theorems to the equation simultaneously. If both leave us in doubt as to the exact number of real roots between $a \cdot 10^n$ and $(a+1)10^n$, in which a is zero or some integer not greater numerically than 9, and n is any integer positive, negative, or zero, apply the process given in Question 1761. Take for example the equation $2x^4 - 9x^3 + 19x^2 - 21x + 8 = 0$.

Fourier's Functions.	*Complementary Functions.*
$2x^4 - 9x^3 + 19x^2 - 21x + 8$	$2x^4 - 9x^3 + 19x^2 - 21x + 8$
$8x^3 - 27x^2 + 38x - 21$	$-9x^3 + 38x^2 - 63x + 32$
$24x^2 - 54x + 38$	$38x^2 - 126x + 96$
$48x - 54$	$-126x + 192$
+	+

Fourier's functions give 4 changes when $x = -\infty$, 4 changes when $x = 0$, 3 changes when $x = 1$, 0 change when $x = 2$, and none when $x = \infty$. The complementary functions give 0 change when $x = -\infty$, 0 change when $x = 0$, 1 change when $x = 1$, 2 changes when $x = 2$, and 4 changes when $x = \infty$. Both theorems, therefore, inform us that no real root exists between $-\infty$ and 0, and only one real root between 0 and 1. The complementary theorem informs us that only one root exists between 1 and 2; and Fourier's theorem informs us that no real root exists greater than 2. The equation, therefore, has only two real roots, one between 0 and 1, and the other between 1 and 2; and each may be approximated to in the usual way by Horner's method.

Next, take the equation $x^4 - 12x^3 + 13x^2 + 24x - 30 = 0$. Either of the theorems will inform us that only one real root exists between $-\infty$ and 0, none between 0 and 1, and only one between 2 and ∞; but both leave us in doubt as to whether two real roots or none exist between 1 and 2. We

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must, therefore, apply the process of Quest. 1761 to separate the two roots if they are real.

We see by Fourier's theorem that $f'(x) = 0$ has one real root between 1 and 2. Denoting this root by x_0 we assume $x_0 = 1 + x_1$ and expand $f(1 + x_1)$ by Horner's method, the result (writing coefficients only) being

$$1 - 8 - 17 + 18 - 4 = f_1(x_1).$$

Our trial fraction for x_1 (since $r = 1$) is $-\frac{A_1}{2A_2}$ that is $\frac{18}{2 \times 17}$, which is between ·5 and ·6. But if we assume $x_1 = ·5 + x_2$ and expand $f_1(·5 + x_2)$ we shall find that A_1 as the coefficient of x_2 is negative, its value 18 as the coefficient of x_1 having been positive. This shows that ·5 is greater than x_1; so to avoid a negative value of x_2 we assume $x_1 = ·4 + x_2$ and expand $f_1(·4 + x_2)$, the result being

$$1 - 6·4 - 25·64 + ·816 - ·0064 = f_2(x_2).$$

Our trial fraction for x_2 is $\frac{·816}{2 \times 25·64}$ which is between ·01 and ·02. We therefore assume $x_2 = ·01 + x_3$ and expand $f_2(·01 + x_3)$, the result being

$$1 - 6·36 - 25·8314 + ·301284 - ·00081039 = f_3(x_3).$$

The trial fraction for x_3 is $\frac{·301284}{2 \times 25·8314}$ which is between ·005 and ·006. We therefore assume $x_3 = ·005 + x_4$ and expand $f_3(·005 + x_4)$. Before completing the expansion, however, we find at the end of the first line of Horner's process, that the absolute term A_0, or the coefficient of x_4^0, is positive, while its value as the absolute term or coefficient of x_3^0, in the former expansion was negative. This shows that a root of $f(x) = 0$ has been passed over, and that, consequently, $f(x) = 0$ has one root between 1·41 and 1·415, and another between 1·415 and 1·42.

If now we remove the restrictions placed upon x_1, x_2, x_3, and make them variables, the equation $f_3(x_3) = 0$ is the equation whose roots are less by 1·41 than those of the original equation $f(x) = 0$; so that if we wish to approximate more closely to either of the two roots which we have just separated, we have only to continue the operation—merely changing the trial fraction from $-\frac{A_1}{2A_2}$ to $-\frac{A_0}{A_1}$. A brief examination of the coefficients A_2, A_1, A_0 suffices to show that the smallest positive root of $f_3(x_3) = 0$ is between ·004 and ·005, and the next between ·007 and ·008; so that the first four figures of the two roots of $f(x) = 0$ are respectively 1·414 and 1·417.

Next, take the equation $x^4 - 12x^3 + 13x^2 + 26x - 33 = 0$. As in the last example, the interval between 1 and 2 is doubtful; so we approximate to 1·45 &c., the root of $f'(x) = 0$, the second expansion $f_2(x_2)$ being $x_2^4 - 6·4x_2^3 - 25·64x_2^2 + 2·816x_2 - ·2064$. The trial fraction $\frac{2·816}{2 \times 25·64}$ shows clearly that x_2 is situated between ·05 and ·06; and since $·06 \times 2·816 - ·2064$ has the same sign as $-·2064$, we are sure, without carrying the process farther, that (Quest. 1761, Art. 8) A_0 will not change its sign in approaching its limit, and that, consequently, the roots between 1 and 2 are imaginary.

Lastly, take the equation $49x^4 - 357x^3 + 1002x^2 - 1305x + 675 = 0$. A doubtful interval exists between 2 and 3, and approximating by the process of Quest. 1761 to 2·14285 &c., the root of $f'(x) = 0$, we find that not only A_1 but apparently also the absolute term A_0 approaches the limit zero. This leads us to suspect that 2·1428 &c. is not only a root of $f'(x) = 0$, but also

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a repeated root of $f(x) = 0$. Applying, therefore, the tests given in Quest. 1761, Art. 13, we find that $49(x-2{\cdot}1428) = 49x-105$ nearly, which, striking out the factor 7, becomes $7x-15$, a factor which will twice divide $f(x)$. Therefore, $2{\cdot}1428\ldots$, or $2\frac{1}{7}$, is a repeated root of $f(x) = 0$.

1812. (Proposed by Professor CAYLEY.)—Find the envelope of a series of circles which touch a given straight line and have their centres in the circumference of a given circle. [See Quest. 1771.]

Solution by the REV. J. L. KITCHIN, M.A.

Let $x^2+y^2 = r^2 \ldots (1)$ be the equation of the given circle, and $x = c$ that of the given line; then the equation of any circle of the system is

$(X-x)^2+(Y-y)^2 = (c-x)^2$, or $X^2-2xX+(Y-y)^2 = c^2-2cx \ldots (2)$;

where x, y are the parameters subject to the condition (1).

Differentiating (1) and (2), and equating the two values of $\frac{dy}{dx}$, we get

$$\frac{X-c}{Y-y} = \frac{x}{y} \ldots (3); \quad \therefore \quad \frac{(X-c)^2+(Y-y)^2}{(X-c)^2} = \frac{r^2}{x^2} = \frac{2(x-c)}{X-c}, \text{ by (2)};$$

$\therefore r^2(X-c) = 2x^3-2cx^2$; whence, putting $x = r\cos\theta, y = r\sin\theta$,

$$X = c+2r\cos^2\theta-2c\cos^2\theta = \tfrac{3}{4}r\cos\theta-c\cos 2\theta+\tfrac{1}{4}r\cos 3\theta \ldots (4).$$

Again, from (3) with use of (4), we get

$Y\cos\theta = r\sin\theta\cos\theta-c\sin\theta+\frac{3}{4}r\sin\theta\cos\theta-c\cos 2\theta\sin\theta+\frac{1}{4}r\cos 3\theta\sin\theta$,

therefore $$Y = \tfrac{3}{4}r\sin\theta-c\sin 2\theta+\tfrac{1}{4}r\sin 3\theta \ldots\ldots\ldots\ldots (5).$$

From (4) and (5), $2X\cos\theta+2Y\sin\theta = 3r+r\cos 2\theta-2c\cos\theta \ldots\ldots (6)$,

also $$(2X-3r\cos\theta)^2+(2Y-3r\sin\theta)^2 = r^2+4c^2-4cr\cos\theta \ldots\ldots (7).$$

From (6) and (7), $(r\cos\theta)^2-\frac{4}{3}c(r\cos\theta) = \frac{1}{3}(X^2+Y^2-r^2-c^2)$,

therefore $r\cos\theta = x = \frac{1}{3}\left\{2c+\sqrt{3(X^2+Y^2-r^2)+c^2}\right\} = \frac{1}{3}(2c+R)$, say;

whence $27r^2(X-c) = 2(2c+R)^3-6c(2c+R)^2 = 2R^3+6cR^2-8c^3$,

therefore $$27r^2(X-c)+2c^3-18c(X^2+Y^2-r^2) = 2R^3,$$

or $$4\left\{3(X^2+Y^2-r^2)+c^2\right\}^3 = \left\{27r^2(X-c)+2c^3-18c(X^2+Y^2-r^2)\right\}^2,$$

the equation to the envelope required, and which is of the sixth degree.

If the whole be orthogonally projected on a plane passing through the axis of x, the given and the variable circles become similar ellipses; hence if a, b be the semi-axes of the given circle so projected into an ellipse, by writing in the last equation x for X and $ab^{-1}y$ for Y, we shall obtain the envelope of a series of ellipses which touch a given straight line at the extremities of their major axes, and have their centres on another ellipse, to which the variable ellipse is similar and similarly placed.

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1814. (Proposed by A. F. TORRY, M.A.)—A triangle is inscribed in an ellipse and envelopes a confocal ellipse: prove that the points of contact lie on the escribed circles.

Solution by E. FITZGERALD; J. DALE; *and others.*

If ABC be the triangle, (the figure is easily imagined,) the tangents B'C', C'A', A'B' to the outer ellipse at A, B, C are the external bisectors of the angles CAB, ABC, BCA; moreover, if A'D, B'E, C'F be drawn perpendicular to BC, CA, AB respectively, D, E, F will be the points of contact of the inscribed confocal ellipse; and D, E, F are obviously the points of contact of the escribed circles with the sides BC, CA, AB.

[See Salmon's *Conics*, 4th ed., Art. 189, and Art. 226, Ex. 3.]

1816. (Proposed by R. BALL, M.A.)—Express the roots of the equation

$$(ae-4bd+3c^2)(ax^4+4bx^3+6cx^2+4dx+e)^2-3\left\{(ac-b^2)x^4+2(ad-bc)x^3\right.$$
$$\left.+(ae+2bd-3c^2)x^2+2(be-cd)x+(ce-d^2)\right\}^2=0,$$

in terms of the roots $\alpha, \beta, \gamma, \delta$ of $x^4+4bx^3+6cx^2+4dx+e=0$.

Solution by PROFESSOR CAYLEY.

Writing

$$U=(a, b, c, d, e)(x, y)^4=a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)$$

$$H'=\left(ac-b^2, \tfrac{1}{2}(ad-bc), \tfrac{1}{6}(ae+2bd-3c^2), \tfrac{1}{2}(be-cd), ce-d^2\right)(x, y)^4$$

$$I=ae-4bd+3c^2;$$

then, considering $x : y$ as the unknown quantity, it is required to find the roots of the equation $IU^2-3H^2=0$ in terms of the roots $(\alpha, \beta, \gamma, \delta)$ of the equation $U=0$; or, what is the same thing, it is required to find the linear factors of the function IU^2-3H^2. The function in question is the product of four quadratic factors, rational functions of $(\alpha, \beta, \gamma, \delta)$; and these being known, the four pairs of linear factors can be determined each of them by the solution of a quadratic equation. In fact, writing

$$\Theta_\alpha=a\left\{(\beta-\alpha)(x-\gamma y)(x-\delta y)+(\gamma-\alpha)(x-\delta y)(x-\beta y)\right.$$
$$\left.+(\delta-\alpha)(x-\beta y)(x-\gamma y)\right\},$$

$$\Theta_\beta=a\left\{(\gamma-\beta)(x-\delta y)(x-\alpha y)+(\delta-\beta)(x-\alpha y)(x-\gamma y)\right.$$
$$\left.+(\alpha-\beta)(x-\gamma y)(x-\delta y)\right\},$$

$$\Theta_\gamma=a\left\{(\delta-\gamma)(x-\alpha y)(x-\beta y)+(\alpha-\gamma)(x-\beta y)(x-\delta y)\right.$$
$$\left.+(\beta-\gamma)(x-\delta y)(x-\alpha y)\right\},$$

$$\Theta=a\left\{(\alpha-\delta)(x-\beta y)(x-\gamma y)+(\beta-\delta)(x-\gamma y)(x-\alpha y)\right.$$
$$\left.+(\gamma-\delta)(x-\alpha y)(x-\beta y)\right\},$$

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we have identically $256(IU^2-3H^2) = \Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$; so that the quadratic factors of IU^2-3H^2 are Θ_α, Θ_β, Θ_γ, Θ_δ. To show that this is so, it is to be remarked that the product $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ is a symmetrical function of the roots $\alpha, \beta, \gamma, \delta$, and consequently a rational and integral function of the coefficients (a, b, c, d, e) of U; moreover Θ_α, Θ_β, Θ_γ, Θ_δ being each of them a covariant (an irrational one) of U, the product in question must be a covariant. But a covariant is completely determined when the leading coefficient is given; hence it will be sufficient to show that the leading coefficients, or coefficients of x^8, in the functions $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ and $256(IU^2-3H^2)$ are equal to each other. Writing for a moment $\Sigma\alpha = p$, $\Sigma\alpha\beta = q$, $\Sigma\alpha\beta\gamma = r$, $\alpha\beta\gamma\delta = s$, the coefficient of x^2 in $\alpha^{-1}\Theta_\alpha$ is $\beta+\gamma+\delta-3\alpha$, which $= p-4\alpha$; we have thence the product $(p-4\alpha)(p-4\beta)(p-4\gamma)(p-4\delta)$, which is $= p^4-4p^3.p+16p^2.q-64p.r+256s$, $= 256s-64pr+16p^2q-3p^4$.

Hence, restoring the omitted factor a^4, and observing that we have $p = -\frac{4b}{a}$, $q = \frac{6c}{a}$, $r = -\frac{4d}{a}$, $s = \frac{e}{a}$, the coefficent of x^8 in $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ is

$256(a^3e-4a^2bd+6ab^2c-3b^4)$, or $256\left\{(ae-4bd+3c^2)a^2-3(ac-b^2)^2\right\}$,

and is consequently equal to the coefficient of x^8 in $256(IU^2-3H^2)$; which proves the theorem.

It may be remarked that the leading coefficient of IU^2-3H^2 is $= a^{-1}(a, b, c, d, e)(b, -a)^4$; and that for a quantic $U = (a, b, ..)(x, y)^n$ of the order n we have a corresponding covariant of the order $n(n-2)$, the leading coefficient of which is $= a^{-1}(a, b, ...)(b, -a)^n$. For $n = 2$, this is the invariant (discriminant) $ac-b^2$; for $n = 3$ it is the cubicovariant $(a^2d-3abc+2b^3, ...)(x, y)^3$; for $n = 4$ it is, as we have seen, the covariant IU^2-3H^2. For $n = 5$, the leading coefficient $a^4f-5a^3be+10a^2b^2d-10ab^3c+4b^5$ is $= a^2(a^2f-5abe+2acd+8b^2d-6bc^2) - 2(ac-b^2)(a^2d-3abc+2b^3)$, which shows that the covariant in question (of the order 15) is $= U^2$ (No. 17) -2 (No. 15) (No. 18), where the Nos. refer to the Tables of my Second Memoir on Quantics, *Phil. Trans.*, vol. 146 (1856), pp. 101—126.

[The roots of $\Theta_\alpha = 0$ are readily found to be

$$\frac{\alpha(\beta+\gamma+\delta)-(\gamma\delta+\delta\beta+\beta\gamma) \pm \left\{\frac{1}{3}\left[(\alpha-\beta)^2(\gamma-\delta)^2+(\alpha-\gamma)^2(\delta-\beta)^2+(\alpha-\delta)^2(\beta-\gamma)^2\right]\right\}^{\frac{1}{2}}}{3\alpha-(\beta+\gamma+\delta)}$$

these then, with three similar pairs, express the eight roots as required.]

1823. (Proposed by W. K. CLIFFORD.)—The conicoids which pass through six fixed points in space, intersect any plane in a series of conics having a common self-conjugate quadrilateral. *Any four* conics have a common self-conjugate quadrilateral. (DEF.—A quadrilateral is *self-conjugate* in respect of a conic which divides its diagonals harmonically.)

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Solution by PROFESSOR CREMONA.

On donne, dans un plan, quatre coniques (1), (2), (3), (4). Les trois premières déterminent un réseau, dont la courbe Hessienne (du 3e ordre) est le lieu des couples de points conjugués par rapport à ces coniques (1), (2), (3). De même, les coniques (1), (2), (4) donnent une autre cubique. Ces deux cubiques passent évidemment par les sommets du triangle conjugué à (1) et (2); donc elles se couperont en six autres points. Si o est l'un de ces points, les droites polaires de o par rapport aux quatre coniques se rencontreront en un même point o' qui appartiendra par conséquent aux deux cubiques. Donc, les six points communs aux deux cubiques sont conjugués, deux à deux, par rapport aux quatre coniques; et, par suite d'un théorème connu (dû à M. Hesse), ils sont les sommets d'un quadrilatère complet. Ainsi, il y a un quadrilatère dont les diagonales sont divisées harmoniquement par les quatre coniques. De plus, cette propriété se vérifie évidemment pour toutes les coniques liées linéairement aux coniques données.

On peut obtenir un tel système linéaire des coniques

$$(1)+\lambda(2)+\mu(3)+\nu(4)$$

en coupant par un plan un système analogue de surfaces du second ordre, *par ex.*, le système des conicoïdes qui passent par six points donnés, ou bien le système des surfaces polaires des points de l'espace par rapport à une surface cubique, &c.

1828. (Proposed by R. WARREN, B.A.)—Exhibit the expression

$$V \equiv (\beta-\gamma)^2(\beta+\gamma-2\alpha)+(\gamma-\alpha)^2(\gamma+\alpha-2\beta)+(\alpha-\beta)^2(\alpha+\beta-2\gamma)$$

(1) as the product of three factors, (2) as the sum of two cubes.

Solution by W. SPOTTISWOODE, F.R.S.; Rev. J. L. KITCHIN, M.A.; W. H. LAVERTY; H. M. TAYLOR, B.A.; *and the* PROPOSER.

Let $\beta+\gamma-2\alpha=p$, $\gamma+\alpha-2\beta=q$, $\alpha+\beta-2\gamma=r$, then

$$3(\beta-\gamma)=r-q,\quad 3(\gamma-\alpha)=p-r,\quad 3(\alpha-\beta)=q-p,\quad p+q+r=0;$$

therefore $V=\frac{1}{9}\{p(q-r)^2+q(r-p)^2+r(p-q)^2\}$

$$=\tfrac{1}{9}\{qr(q+r)+rp(r+p)+pq(p+q)-6pqr\}=-pqr.$$

Moreover the developed form of the expression, viz.,

$$V=2(\alpha^3+\beta^3+\gamma^3)-3(\beta^2\gamma+\beta\gamma^2+\gamma^2\alpha+\gamma\alpha^2+\alpha^2\beta+\alpha\beta^2)-12\alpha\beta\gamma,$$

shows almost by inspection that the form of the two cubes must be

$$(\alpha+\omega\beta+\omega^2\gamma)^3+(\alpha+\omega^2\beta+\omega\gamma)^3,$$

where ω is an imaginary cube root of unity.

[Since $p^3+q^3+r^3-3pqr\equiv\Sigma p(\Sigma p^2-\Sigma qr)=0$, $\because \Sigma p=0$, we have $V=-pqr=-\frac{1}{3}(p^3+q^3+r^3)$, which exhibits V as the sum of *three* cubes.]

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1847. (Proposed by W. S. BURNSIDE, B.A.)—Prove that the method given by Tschirnhausen for removing the second and fourth terms from the binary quartic $(a, b, c, d, e)(x, y)^4 = 0$ depends ultimately on the solution of the equation $\lambda^3 - I\lambda + 2J = 0$,
where $I = ae - 4bd + 3c^2$, and $J = ace + 2bcd - ad^2 - eb^2 - c^3$.

Solution by ROBERT BALL, M.A.

Removing the second term from $(a, b, c, d, e)(x, 1)^4 = 0$, it becomes (putting $a = 1$), $x^4 - 6Hx^2 + 4Gx + I - 3H^2 = 0$,
where $H = b^2 - ac$, $G = 2b^3 - 3abc + a^2d$, and $G^2 = 4H^3 - IH - J$.

Now the principle of Tschirnhausen's method assumes $y = x^2 + Bx + C$, and x is then eliminated between the equations

$x^4 - 6Hx^2 + 4Gx + I - 3H^2 = 0$, $x^2 + Bx + C - y = 0$; the result being

$$(C-y)^4 + 12H(C-y)^3 + (2I + 30H^2 + 12BG - 6HB^2)(C-y)^2$$
$$+ \left\{28H^3 - 4IH - 16J + 24HGB + (12H^2 - 4I)B^2 - 4GB^3\right\}(C-y)$$
$$+ I^2 - 6IH^2 + 9H^4 + (12GH^3 - 4GI)B + (18H^3 - 6IH)B^2 + (I - 3H^2)B^4 = 0.$$

We must express that the coefficients of y^3 and y vanish; hence
$-4C - 12H = 0$, and $-4C^3 - 36HC^2 - 2C(2I + 30H^2 + 12BG - 6HB^2)$
$-28H^3 + 4IH + 16J - 24HGB + (4I - 12H^2)B^2 + 4GB^3 = 0$.

From the first we obtain $C = -3H$, and this substituted in the second gives $GB^3 + B^2(I - 12H^2) + 12HGB - 4G^2 = 0$; and if in this last equation we change B into $\frac{2G}{H + \lambda}$, it becomes $\lambda^3 - I\lambda + 2J = 0$.

NOTE.—The more general investigation for the case when the elimination is performed between a cubic and a biquadratic will shortly appear in the *Quarterly Journal of Mathematics*, and various reducing cubics resulting from the process will be reduced to the form $x^3 - Ix + 2J = 0$. The present Question will be also presented in a different form.

1811. (Proposed by Professor SYLVESTER.)—(α). Prove that the chance of three points taken at random within a circle or sphere forming the vertices of a triangle of any prescribed form is the same as if two of the points are restrained to move on the circumference or surface, with a likelihood varying as the square of the distance between them.

(β). By the aid of this principle show that the chance of a triangle whose angular points are taken at random within a circle being acute is

$$\int_0^{\frac{1}{2}\pi} (3\sin^3\theta\cos\theta + \theta\sin^2\theta - \tfrac{1}{2}\pi\sin^4\theta)\,d\theta + \int_0^{\frac{1}{2}\pi} \pi\sin^2\theta\,d\theta, \text{ or } \frac{4}{\pi^2} - \frac{1}{8};$$

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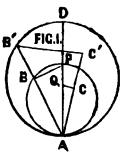
and that the chance of a triangle whose angular points are taken at random within a sphere being obtuse is

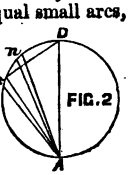
$$\int_0^{\frac{1}{2}\pi} \sin^3\theta \cos\theta\,(6-6\sin\theta-3\cos\theta+4\sin^2\theta+\cos^3\theta)\,d\theta, \quad \text{or} \quad \frac{37}{70}.$$

(γ). Find also the chance that no angle of the triangle will exceed any magnitude α between $\frac{1}{3}\pi$ and π, distinguishing between the case where α is greater and that where it is less than $\frac{1}{2}\pi$.

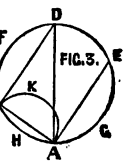
Solution by J. D. EVERETT, D.C.L.

1. Of the three points A, B, C, let A be that which is farthest from the centre of the circle. Then A may (as shown in the Solution of Quest. 1333), be made a fixed point on the circumference. To AB, AC (Fig. 1) let perpendiculars BP, CQ be drawn, meeting the diameter AD in P, Q; and let AP be greater than AQ. Then if a circle be drawn on AP as diameter, B will be on its circumference, and C within it. Hence there is no loss of generality in restricting C to lie within the variable circle which passes through A, B, and has its centre in AD. If AB be produced to meet the circumference of the given circle in B′, and AC be produced, in the same ratio, to C′, the triangle AB′C′ will be similar to ABC. Hence B may be restricted to lie on the circumference provided allowance be made for the relative probability of different directions of AB. Now if m and n (Fig. 2) be two equal small arcs, and if sectors Am, An be formed by joining A to their extremities, the relative probabilities of B being within either of these sectors are as the areas of the sectors, that is ultimately as the squares of the distances Am, An. Hence, in restricting B to lie on the circumference, we must make the likelihood vary as the square of the distance from A. This proves the theorem (α).





2. To find the probability of the triangle ABC being acute for any given position of B on the circumference, draw (Fig. 3) chords AE, BD perpendicular to AB; and on AB describe the semicircle AKB on the side next the centre. The triangle will be acute if C lies within the portion AKBDE of the circle, and obtuse if it lies in any other position. Hence the probability of an acute triangle is (AKBDE ÷ circle). Let θ denote the angle BDA; then, taking the radius of the circle for unit of length, the area of AKBDE is $3\sin\theta\cos\theta+\theta-\frac{1}{2}\pi\sin^2\theta$, and the area of the circle is π; hence, for a given position of B, the probability of an acute triangle $(3\sin\theta\cos\theta+\theta-\frac{1}{2}\pi\sin^2\theta)\div\pi$. But the likelihood of any value of θ is as the square of AB, that is, as $\sin^2\theta$; hence the chance of the triangle being acute is



$$\int_0^{\frac{1}{2}\pi} (3\sin^3\theta\cos\theta+\theta\sin^2\theta-\tfrac{1}{2}\pi\sin^4\theta)\,d\theta \div \int_0^{\frac{1}{2}\pi} \pi\sin^2\theta\,d\theta = \frac{4}{\pi^2}-\frac{1}{8}.$$

The integrals are easily evaluated, and need no remark.

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3. In adapting the foregoing reasoning to a sphere, BP and CQ (Fig. 1) will represent lines perpendicular to AB, AC, meeting the diameter AD of the sphere in P, Q; then the sphere whose diameter is AP will have B on its surface and C within it, and it may consequently be shown as before that B may be restricted to lie on the surface according to a certain law of likelihood. To find this law, let m (Fig. 2) denote the area of a small portion of the surface, then the volume of the cone whose base is m and vertex A is $\frac{1}{3}mh \sin\theta$, h denoting the distance Am and θ the angle ADm which is equal to the obliquity of the cone or the inclination of Am to a tangent plane at m. But $\sin\theta$ evidently varies as h; hence if equal small portions m be taken on different parts of the surface, the likelihood of B being in any one of them must vary as $\sin^2\theta$ or as h^2, that is, as the square of the distance from A. To find the probability of an acute triangle for any given position of B, draw (Fig. 3) through A, B planes AE, BD perpendicular to AB, and through AB draw a plane perpendicular to the plane ABDE; also on the section of the sphere made by the plane last drawn as base, describe the hemisphere AKB. The triangle will be acute if C be within the portion AKBDE of the sphere contained between the hemisphere AKB and the two planes AE, BD; and it will be obtuse if C lies in one of the other portions. Hence the chance of an obtuse triangle is the sum of these other portions divided by the volume of the sphere. Let θ denote the angle ADB; then (the radius of the sphere being unity) the volumes of the segment AHB; of the two equal segments BFD, AGE; of the hemisphere AKB; and of the whole sphere; are, respectively,

$$\tfrac{1}{3}\pi(2-3\cos\theta+\cos^3\theta),\ \tfrac{2}{3}\pi(2-3\sin\theta+\sin^3\theta),\ \tfrac{2}{3}\pi\sin^3\theta,\ \tfrac{4}{3}\pi;$$

hence, for a given position of B, the chance of an obtuse triangle is

$$\tfrac{1}{3}\pi\left\{(4-6\sin\theta+2\sin^3\theta)+(2-3\cos\theta+\cos^3\theta)+2\sin^3\theta\right\} \div \tfrac{4}{3}\pi.$$

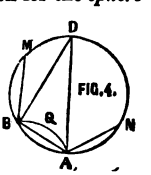
In supposing B to move over the surface, imagine the surface to be divided into narrow zones of equal width, having A as their common pole. Then the area of any zone will be proportional to its radius which is $2\sin\theta\cos\theta$; hence if all situations on the sphere were equally probable, the likelihood of any value of θ would vary as $\sin\theta\cos\theta$; and since the likelihood of different situations varies as $\sin^2\theta$, it follows by compounding ratios that the likelihood of any value of θ is as $\sin^3\theta\cos\theta$; hence, in this case, the chance of an obtuse triangle is

$$\frac{1}{4}\int_0^{\frac{1}{2}\pi}\sin^3\theta\cos\theta\,(6-6\sin\theta-3\cos\theta+4\sin^3\theta+\cos^3\theta)\,d\theta+\int_0^{\frac{1}{2}\pi}\sin^3\theta\cos\theta\,d\theta,$$

and since the divisor here is equal to $\frac{1}{4}$, the required chance becomes

$$\int_0^{\frac{1}{2}\pi}\sin^3\theta\cos\theta\,(6-6\sin\theta-3\cos\theta+4\sin^3\theta+\cos^3\theta)\,d\theta=\frac{37}{70}.$$

[4. To solve the problem (γ) for the *circle* (and the solution for the *sphere* may be obtained in the same way) we may proceed as follows. Construct Figs. 4, 5, 6, so that $\alpha = \angle ABM = BAN = \angle$ in seg. AQB. Then if $\alpha > \frac{1}{2}\pi$, Fig. 4 applies; and no angle of the triangle will exceed α when C falls within the area ANMBQA ($=\Delta_1$ suppose); moreover, when $\theta > \pi-\alpha$ the lines AN, BM lie outside the segment ADB, and then the like *area locus* of C is $\pi-2$ seg. AQB ($=\Delta_2$ suppose). Now putting, for shortness' sake F (ϕ) to denote the area of a circular segment,



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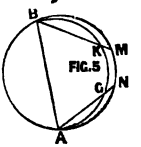
to radius unity, which contains an angle ϕ, so that $F(\phi) \equiv \pi - \phi + \sin\phi\cos\phi$, we readily find radius of arc AQB $= \sin\theta \operatorname{cosec}\alpha$ $(= r$ suppose), also $\Delta_1 = F(\theta) - 2F(\alpha+\theta) - r^2F(\alpha)$, and $\Delta_2 = \pi - 2r^2F(\alpha)$.

Furthermore, for an assigned direction of AB (that is to say, for each value of θ) the measure of the number of points B projected to the circumference in Fig. 1 is $\sin^2\theta$. Hence the whole probability $p(\alpha)$ that no angle of the triangle exceeds a magnitude α greater than $\frac{1}{2}\pi$, is

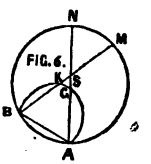
$$p(\alpha) = \left\{\int_0^{\pi-\alpha} \Delta_1 \sin^2\theta\, d\theta + \int_{\pi-\alpha}^{\pi} \Delta_2 \sin^2\theta d\theta\right\} \div \int_0^{\frac{1}{2}\pi} \pi \sin^2\theta d\theta$$

$$= \frac{1}{\pi^2}\left\{\pi^2 - 3(\pi-\alpha)^2 + \sin^2\alpha(\sin^2\alpha + 3) - \right.$$

$$\left. \tfrac{8}{3}\operatorname{cosec}^2\alpha\, F(\alpha)\, F(\pi-\alpha) + \sin\alpha\cos\alpha\, F(\alpha)\right\}.$$



5. When $\alpha < \frac{1}{2}\pi$, no angle of the triangle will exceed α if C fall within the area GKS $(= \Delta_3)$ in Fig. 6, or GKMN $(= \Delta_4)$ in Fig. 5; moreover Fig. 6 applies till S falls on the given circle, that is from $\theta = 0$ to $\theta = \pi - 2\alpha$; also Fig. 5 applies from $\theta = \pi - 2\alpha$ to $\theta = \alpha$, when the arc AGKB coincides with the given circle; and from $\theta = \alpha$ to $\theta = \frac{1}{2}\pi$ no such triangle can be formed.



Also $\Delta_3 = \tan\alpha \sin^2\theta - r^2\left\{F(\alpha) - 2F(2\alpha)\right\}$,

and $\Delta_4 = \left\{F(\theta) - 2F(\alpha+\theta)\right\} - r^2\left\{F(\alpha) - 2F(2\alpha)\right\}$.

Hence the probability that no angle of the triangle exceeds a magnitude α less than $\frac{1}{2}\pi$, is

$$p(\alpha) = \left\{\int_0^{\pi-2\alpha} \Delta_3 \sin^2\theta d\theta + \int_{\pi-2\alpha}^{\alpha} \Delta_4 \sin^2\theta d\theta\right\} \div \int_0^{\frac{1}{2}\pi} \pi \sin^2\theta d\theta$$

$$= \frac{1}{\pi^2}\left\{(3\alpha-\pi)^2 + \sin^2\alpha(\sin^2\alpha+3) - \sin^2 2\alpha(\sin^2 2\alpha + 3)\right.$$

$$+ (\tfrac{8}{3}\pi - 3\alpha + \sin 4\alpha - \tfrac{1}{8}\sin 8\alpha)\tan\alpha$$

$$\left. + (\pi - 3\alpha - \tfrac{1}{2}\sin 2\alpha + \sin 4\alpha)(\tfrac{8}{3}\alpha - \sin 2\alpha + \tfrac{1}{8}\sin 4\alpha)\operatorname{cosec}^2\alpha\right\}.$$

6. Using the expression in Art. 5 for $\alpha = \frac{1}{3}\pi$, that in Art. 4 for $\alpha = \frac{2}{3}\pi$ and $\alpha = \pi$, and either of the two for $\alpha = \frac{1}{2}\pi$, we obtain

$$p(\tfrac{1}{3}\pi) = 0,\ p(\tfrac{1}{2}\pi) = \frac{4}{\pi^2} - \frac{1}{8},\ p(\tfrac{2}{3}\pi) = \frac{27}{8\pi^2} + \frac{\sqrt{3}}{12\pi} + \frac{2}{9},\ p(\pi) = 1.$$

Thus $p(\frac{1}{2}\pi) = \cdot2803$ nearly, and $p(\frac{2}{3}\pi) = \cdot6101$ nearly.]

1754. (Proposed by T. COTTERILL, M.A.)—If a triangle and its nine-point circle are drawn; in the parabola which touches two sides and the perpendiculars to them through the opposite angles, the focus, the directrix, and the tangents to it from the centre of the circle are given by the figure.

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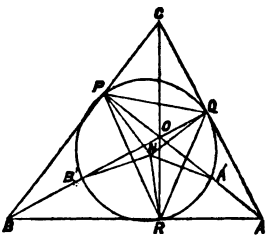
Solution by R. WARREN, B.A.; J. DALE; E. FITZGERALD; *and others.*



Let the parabola touch the sides CA, CB, and the perpendiculars AP, BQ; and let N be the centre of the nine-point circle. Then, since the circle circumscribing the triangle formed by any three tangents to a parabola, passes through the focus, the point R is the *focus* of the parabola. Again, since the locus of the intersection of rectangular tangents to a parabola is the directrix, the line PQ is the directrix of the parabola. Moreover, if the nine-point circle meet OA, OB in A', B' respectively, NA' and NB' are tangents to the parabola; since, as is easily proved, NA', NB' are respectively perpendicular to RQ, RP.

1753. (Proposed by Dr. BOOTH, F.R.S.)—The tangential equation of the surface of the centres of curvature of an ellipsoid being

$$(\xi^2+v^2+\zeta^2)^2 = \left(\frac{\xi^2}{a^2}+\frac{v^2}{b^2}+\frac{\zeta^2}{c^2}\right)(a^2\xi^2+b^2v+c^2\zeta^2-1);$$

show that any two parallel tangent planes being drawn to the surface of centres and to the ellipsoid, the difference of the squares of the coincident perpendiculars let fall upon them from the common centre is always equal to the square of the coinciding semidiameter of the ellipsoid.

Solution by W. S. BURNSIDE, B.A.

Comparing the equations $\xi x+vy+\zeta z=1$, and $x\cos\alpha+y\cos\beta+z\cos\gamma=p$ we have $p\xi=\cos\alpha$, $pv=\cos\beta$, $p\zeta=\cos\gamma$. Now, substituting these values of ξ, v, ζ in the given equation of the surface of centres, we have $r^2=\pi^2-p^2$, where $\frac{1}{r^2}=\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}$, and $\pi^2=a^2\cos^2\alpha+b^2\sin\beta+c^2\cos^2\gamma$; but π and r are the perpendicular on the tangent plane and the coincident radius vector of the ellipsoid; hence the relation in the Question.

1795. (Proposed by W. K. CLIFFORD.)—(1) Let P be the point in a homogeneous triangular lamina ABC at which the sides subtend equal angles. Show that if the lamina be placed in a smooth prolate spheroid whose long axis is vertical, it will rest in equilibrium when the point P coincides with the lower focus of the spheroid.

(2) If the lamina be not homogeneous, and its centre of gravity be given, construct for the corresponding position of the point P.

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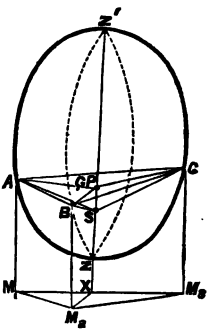
Solution by the Rev. J. L. Kitchin, M.A.



1. Let ABC be the triangle in the spheroid, ZZ' the axis, M_1XM_3 the directrix of its generating ellipse. Let AM_1, BM_2, CM_3 be perpendicular to the plane described by the line M_1X revolving with the generating ellipse; and let e be the eccentricity and S the common focus of the sections. Join AS, BS, CS. Then the height of the centre of gravity (G) of the triangle above the plane $M_1XM_2 = \frac{1}{3}(AM_1 + BM_2 + CM_3) = \frac{1}{3}e^{-1}(AS + BS + CS)$, and this must be a minimum. Now $AS + BS + CS$ is made a minimum, first, by S being a point in the triangle ABC, and secondly, by its coinciding with the point P whose position is determined by subtending equal angles with the sides. Whence the position of equilibrium.

2. In the second case, the height of G above the plane $M_1XM_2 = m \cdot AM + n \cdot BM_2 + p \cdot CM_3 = e^{-1}(m \cdot AS + n \cdot BS + p \cdot CS)$, where m, n, p are constants depending upon the position of G; and this will be a minimum, first by S being at a point (P) in the triangle, and secondly by $m \cdot AP + n \cdot BP + p \cdot CP$ being a minimum. We have then this problem, "to find a point in a triangle such that the sum of its distances from the angular points, multiplied by given constants, may be a minimum;" a problem which has been solved by Fuss in the *Nova Acta Petrop.*, by Bertrand in *Liouville's Journal*, by Wallace in his *Theorems and Formulæ*, p. 120, and by many others. The point P so determined is the point where S lies in the triangle in the position of equilibrium.

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